

Long time semiclassical Egorov theorem for \hbar -pseudodifferential systems

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Abstract

In the Heisenberg picture, we study the semiclassical time evolution of a bounded quantum observable $Q^w(x, \hbar D_x; \hbar)$ associated to a $(m \times m)$ matrix-valued symbol Q generated by a semiclassical matrix-valued Hamiltonian $H \sim H_0 + \hbar H_1$. Under a non-crossing assumption on the eigenvalues of the principal symbol H_0 that ensures the existence of almost invariant subspaces of $L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$, and for a class of observables that are semiclassically block-diagonal with respect to the projections onto these almost invariants subspaces, we establish a long time matrix-valued version for the semiclassical Egorov theorem valid in a large time interval of Ehrenfest type $T(\hbar) \simeq \log(\hbar^{-1})$.

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1 Introduction

Known as Bohr's correspondence principle in physics and Egorov theorem in mathematical literature, the semiclassical approximation ensures the transition between quantum and classical evolutions of observables. The relation between the classical and quantum evolutions may be considered as one of the oldest problems of semiclassical analysis. The starting point was the famous Bohr's correspondence principle proposed by Niels Bohr on 1923. According to this principle, the quantum evolution of an observable is closer and closer to its classical evolution as the Planck constant \hbar becomes negligible. For many years, the semiclassical approximation has been the object of several investigations following two approaches. The first one uses semiclassical wave packets (or coherent states) as initial data and aims to approximate the evolved wave packet by a linear combination of coherent states (see [29], [17], [18], [8] and [9] for a complete description of coherent states). The second approach, that will be our principal subject in this paper, considers the Heisenberg evolution of suitable bounded observables and seeks to construct an approximation for the corresponding observables in terms of \hbar -pseudodifferential operators (see [33], [30], [6], [20]). Let us start by recalling the general setting of this approach.

Let $H \in C^\infty(\mathbb{R}^{2n}; \mathbb{R})$ be a classical Hamiltonian and $Q \in C^\infty(\mathbb{R}^{2n}; \mathbb{R})$ be a real observable. We consider the system of Hamilton equations

$$\frac{dx}{dt} = \partial_\xi H(x, \xi), \quad \frac{d\xi}{dt} = -\partial_x H(x, \xi). \quad (1.1)$$

These equations generate a flow ϕ_H^t called the Hamiltonian flow associated to H defined by $\phi_H^t(x = x(0), \xi = \xi(0)) = (x(t), \xi(t))$ and $\phi_H^t|_{t=0}(x, \xi) = (x, \xi)$. In the sequel, additional assumption will be required on H so that for all initial data $(x, \xi) \in \mathbb{R}^{2n}$, $\phi_H^t(x, \xi)$ is well defined for all $t \in \mathbb{R}$.

The time evolution of Q under the flow ϕ_H^t given by $q_0(t) := Q \circ \phi_H^t$ is described by the equation

$$\frac{d}{dt}q_0(t) = \{H, Q\} \circ \phi_H^t, \quad q_0(t)|_{t=0} = Q, \quad (1.2)$$

where $\{H, Q\}$ denotes the Poisson bracket of H, Q defined by

$$\{H, Q\} := \partial_\xi H \cdot \partial_x Q - \partial_x H \cdot \partial_\xi Q.$$

Let $H^w := H^w(x, \hbar D_x)$ and $Q^w := Q^w(x, \hbar D_x)$ be the self-adjoint operators in $L^2(\mathbb{R}^n)$ associated to H and Q , respectively. Here we use the \hbar -Weyl quantization (see formula (2.1)). Let $e^{-\frac{it}{\hbar}H^w}$ be the unitary operator solution of the evolution equation

$$i\hbar \partial_t u(t) = H^w u(t).$$

The time evolution of the quantum observable Q^w under $e^{-\frac{it}{\hbar}H^w}$ given by $Q(t) := e^{\frac{it}{\hbar}H^w} Q^w e^{-\frac{it}{\hbar}H^w}$ is described by the quantum analogous of (1.2) called the Heisenberg equation of motion

$$\frac{d}{dt}Q(t) = \frac{i}{\hbar}[H^w, Q(t)], \quad Q(t)|_{t=0} = Q^w, \quad (1.3)$$

where $[H^w, Q(t)] := H^w Q(t) - Q(t) H^w$ denotes the commutator of $H^w, Q(t)$.

Under suitable growth assumptions on H , the semiclassical Egorov theorem (see [6, Theorem 1.2] for the precise statement) states that for every fixed t , $Q(t)$ is an \hbar -pseudodifferential operator with principal symbol $q_0(t, x, \xi) = Q \circ \phi_H^t(x, \xi)$. More precisely, there exists a family of symbols $(q_j(t, \cdot, \cdot))_{j \geq 1}$ with $\text{supp}(q_j(t, \cdot, \cdot)) \subset \phi_H^{-t}(K)$, where K is the union of the supports of the q_j , such that for every finite time $\bar{t} > 0$ and for all $N \in \mathbb{N}$, the following estimate

$$\left\| Q(t) - \sum_{j=0}^N \hbar^j (q_j(t))^w \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_N \hbar^{N+1}, \quad (1.4)$$

holds uniformly for $|t| \leq \bar{t}$. The time-dependent family of operators $((q_j(t))^w)_{j \geq 0}$ is called semiclassical approximation of $Q(t)$. The construction of such approximation is obtained in a formal way by solving the Heisenberg equation (1.3) in the space of formal power series of \hbar . If one assumes that $Q(t)$ is an \hbar -pseudodifferential operator with Weyl symbol $q(t)$, where for now we look at $q(t)$ as a formal power series of \hbar , $q(t) \sim \sum_{j \geq 0} \hbar^j q_j(t)$, by (1.3) it fulfils the equation

$$\frac{d}{dt}q(t) = \frac{i}{\hbar}[H, q(t)]_\#, \quad q(t)|_{t=0} = Q, \quad (1.5)$$

where $[H, q(t)]_\# := H \# q(t) - q(t) \# H$, with $H \# q(t)$ denotes the Moyal product of $H, q(t)$ defined as the symbol of $H^w \circ Q(t)$ (see appendix A).

By expanding the symbol $\frac{i}{\hbar}[H, q(t)]_\#$ in powers of \hbar and then equating equal powers of \hbar in both sides, the Cauchy problem (1.5) implies similar Cauchy problems for the symbols $q_j(t)$ given by

$$(\mathcal{C}_j) \begin{cases} \frac{d}{dt}q_j(t) &= \left(\frac{i}{\hbar}[H, q(t)]_\# \right)_j, \\ q_0(t)|_{t=0} &= Q, \quad q_j(t)|_{t=0} = 0, \forall j \geq 1, \end{cases} \quad (1.6)$$

where $\left(\frac{i}{\hbar}[H, q(t)]_\# \right)_j$ is the j -th term (i.e. the coefficient of \hbar^j) in the asymptotic expansion of $\frac{i}{\hbar}[H, q(t)]_\#$.

For $j = 0$, the principal symbol of $[H, q(t)]_\#$ given by the commutator $[H, q_0(t)]$ vanishes. Therefore, the term of order \hbar^0 in the asymptotic expansion of $\frac{i}{\hbar}[H, q(t)]_\#$ coincides with the sub-principal symbol of $i[H, q(t)]_\#$ which is equal to $\{H_0, q_0(t)\}$. Consequently, (\mathcal{C}_0) reads

$$\frac{d}{dt}q_0(t) = \{H, q_0(t)\}, \quad q_0(t)|_{t=0} = Q,$$

and we get $q_0(t) = Q \circ \phi_H^t$. The symbols $q_j(t)$ for $j \geq 1$ are obtained in the same manner by following the above algorithm (see Remark 3.1 for the general formula of the symbols $q_j(t)$).

The semiclassical approximation given by the semiclassical Egorov theorem is limited to the evolutions in finite time intervals. Several works were devoted to the investigation of its validity for large times which may depend on \hbar (see [8] for the time evolution of coherent states, and [1], [6] for the time evolution of observables). These investigations were based on a conjecture going back to the physicists Chirikov and Zaslavski ([7],[34]) which claim that the semiclassical approximation remains valid in a large time interval of length $T(\hbar) \simeq \log(\hbar^{-1})$ known as the Ehrenfest time. The optimal time for which this claim was proved has been obtained by Bouzouina and Robert [6]. They showed that the L^2 -operator norm of the remainder term in the asymptotic expansion of $Q(t)$ given by the left hand side of (1.4) is uniformly dominated, at any order, by an exponential term whose argument is linear in time. In particular, this allows them to recover the Ehrenfest time for the validity of the semiclassical approximation.

The purpose of this paper is to study the extension of the above results to the case of matrix-valued observables. Given a bounded quantum observable $Q^w(x, \hbar D_x; \hbar)$ associated to a $(m \times m)$ matrix-valued symbol $Q(x, \xi; \hbar) \sim \sum_{j \geq 0} \hbar^j Q_j(x, \xi)$, we study the time evolution $Q(t) := e^{\frac{it}{\hbar} H^w} Q^w e^{-\frac{it}{\hbar} H^w}$ generated by a semiclassical $(m \times m)$ hermitian-valued Hamiltonian $H(x, \xi; \hbar)$. We establish a long time matrix-valued version for the semiclassical Egorov theorem by giving a semiclassical approximation for $Q(t)$ valid in a large time interval of Ehrenfest type. To the best of our knowledge, there are no results concerning the large time behaviour of the semiclassical approximation in the matricial case.

Matrix-valued version for Egorov's theorem has been discussed several times in the literature ([11], [28], [5], [3], [20]). Brummelhuis and Nourrigat [5] have studied the particular case of matrix-valued Hamiltonian with scalar principal symbol and proved an extension of the semiclassical Egorov theorem valid for evolutions in finite time intervals. This result has been extended to the general case by Bolte and Glaser [3] under an assumption on the gap between the eigenvalues of the principal symbol of the Hamiltonian (see assumption **(A1)** in the next section). However, their result is again only valid for finite time. Here we require the same assumption as in [3] and we are concerned with the large time behaviour of the approximation. Some ideas from [3] are still present here.

Let us explain the main difficulties arising from the matrix structure of the problem. By going back to the Cauchy problem (\mathcal{C}_0) satisfied by the principal symbol $q_0(t)$, one immediately sees that in the case where H and Q are matrix-valued functions, the principal symbol of the Moyal commutator $[H, q(t)]_\#$ which is equal to the matrix commutator $[H_0, q_0(t)]$ is no longer zero. Here H_0 denotes the principal symbol of H . Then, at leading semiclassical order, we have an equation of the type

$$\frac{d}{dt} q_0(t) = \frac{i}{\hbar} [H_0, q_0(t)] + \mathcal{O}(\hbar^0), \quad \hbar \searrow 0. \quad (1.7)$$

In order to get a solvable equation for $q_0(t)$, the factor \hbar^{-1} forces us to restrict ourselves to a class of observables for which the commutativity between H_0 and $q_0(t)$ is preserved under the time evolution. For $t = 0$, this is equivalent to a block-diagonal form of Q_0 with respect to the eigenprojectors of H_0 . Under a non-crossing assumption on the eigenvalues of H_0 , we use an idea due to Helffer and Sjöstrand [19] which consists in decomposing the Hilbert space $L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$ into almost invariant subspaces with respect to the time evolution generated by H^w . By considering a class of observables that are block-diagonal with respect to the projections onto these almost invariant subspaces, we reduce the study of $Q(t)$ to that of a family of block-diagonal Heisenberg observables for each of them we construct a formal asymptotic expansion in powers of \hbar by solving the corresponding symbolic Heisenberg problem. This reduction is modulo $\mathcal{O}(\hbar^\infty)$ in $\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)$ uniformly in time in large time intervals which cover the Ehrenfest time. Then, to justify this asymptotic expansion, we control the remainder term at any order by giving a uniform exponential estimate with linear argument in time. In particular, this estimate allows us to recover the Ehrenfest time for the validity of the semiclassical approximation.

Another difficulty related to the matrix structure of H and Q lies in the fact that the time evolution of the symbols of the constructed approximation will be governed not only by the Hamiltonian flows (generated by the eigenvalues of H_0), but also by a conjugation by a family of transport matrices that we

have to control the behaviour of their derivatives uniformly in time.

The paper is organised as follows. In section 2, after introducing the classes of symbols that we shall use through the paper, we state our main results. Section 3 will be devoted to the study of the particular case of Hamiltonian with scalar principal symbol (and matrix-valued sub-principal symbol). In section 4, we generalize the results of section 3 to the case of matrix-valued principal symbol without crossing eigenvalues. The appendix A contains a short background on some basic results of \hbar -pseudodifferential calculus in the context of operators with matrix-valued symbols. In appendices B and C, we give the proofs of some technical results.

Some notations : Let $M_m(\mathbb{C})$ be the space of $(m \times m)$ complex-valued matrices endowed with the operator norm denoted by $\|\cdot\|$. We denote I_m the corresponding identity matrix.

In this paper, three types of commutators appear : for P, Q two matrix-valued functions in some suitable classes of symbols, $[P, Q] := PQ - QP$ is the usual matrix commutator. We use the same notation for the standard operators commutator $[P^w, Q^w] := P^w Q^w - Q^w P^w$. Finally, the symbol of $[P^w, Q^w]$ will be denoted $[P, Q]_\# := P\#Q - Q\#P$ and called the Moyal commutator of P and Q .

Through the paper smooth means C^∞ . For $A \in C^\infty(\mathbb{R}^{2n}) \otimes M_m(\mathbb{C})$ and $\alpha, \beta \in \mathbb{N}^n$, we introduce the notation

$$A_{(\beta)}^{(\alpha)}(x, \xi) := \partial_\xi^\beta \partial_x^\alpha A(x, \xi).$$

Given a function f_\hbar depending on the semiclassical parameter $\hbar \in (0, 1]$, the asymptotic relation $f_\hbar = \mathcal{O}(\hbar^\infty)$ means that $f_\hbar = \mathcal{O}(\hbar^N)$, for all $N \in \mathbb{N}$.

The identity operator on $L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$ will be denoted $\text{id}_{L^2(\mathbb{R}^n) \otimes \mathbb{C}^m}$. For $\zeta = (\zeta_1, \dots, \zeta_{2n}) \in \mathbb{R}^{2n}$, we use the standard notation $\langle \zeta \rangle := (1 + |\zeta|^2)^{\frac{1}{2}} = (1 + |\zeta_1|^2 + \dots + |\zeta_{2n}|^2)^{\frac{1}{2}}$. Finally, our convention for the Poisson bracket of matrix-valued functions $A, B \in C^\infty(\mathbb{R}^{2n}) \otimes M_m(\mathbb{C})$ is

$$\{A, B\} := \partial_\xi A \partial_x B - \partial_x A \partial_\xi B.$$

Notice that in general $\{A, B\} \neq -\{B, A\}$.

2 Assumptions and main results

Let us begin by recalling some notions about semiclassical classes of symbols. We refer to [12, ch. 7] and [35, ch. 4] for more details. For the context of operators with matrix-valued symbols see [20, ch. 1].

In this paper we use the standard \hbar -Weyl quantization defined for $A \in \mathcal{S}(\mathbb{R}^{2n}) \otimes M_m(\mathbb{C})$ (the space of Schwartz functions on \mathbb{R}^{2n} with values in $M_m(\mathbb{C})$) by the formula

$$A^w(x, \hbar D_x)u(x) := \frac{1}{(2\pi\hbar)^n} \int \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\langle x-y, \xi \rangle} A\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}^m. \quad (2.1)$$

For short, we shall sometimes simply write A^w .

Let $g : \mathbb{R}^{2n} \rightarrow [1, +\infty[$ be an order function, i.e. g satisfies : there exists $C, N > 0$ such that

$$g(v) \leq C \langle v - w \rangle^N g(w), \quad \forall v, w \in \mathbb{R}^{2n}.$$

The typical example is $g(x, \xi) = \langle (x, \xi) \rangle^a$, $a \geq 0$.

Definition 2.1 (i) We denote by $S(g; \mathbb{R}^{2n}, M_m(\mathbb{C}))$ the set of smooth functions on \mathbb{R}^{2n} with values in $M_m(\mathbb{C})$ satisfying : for all multi-index $\gamma \in \mathbb{N}^{2n}$, there exists a constant $C_\gamma > 0$ such that for all $(x, \xi) \in \mathbb{R}^{2n}$,

$$\|\partial_{(x, \xi)}^\gamma A(x, \xi)\| \leq C_\gamma g(x, \xi). \quad (2.2)$$

We will write it simply $S(g)$ when no confusion can arise. Symbols in $S(g)$ may depend on the semiclassical parameter $\hbar \in (0, 1]$. In this case, we say that $A \in S(g)$ if $A(\cdot, \cdot; \hbar)$ is uniformly bounded in $S(g)$ when \hbar varies in $(0, 1]$.

For $r \in \mathbb{R}$, we define the classes

$$S^r(g) := \hbar^{-r} S(g), \quad S^{-\infty}(g) := \bigcap_{r \in \mathbb{R}} S^r(g).$$

- (ii) $A \in S(g)$ is said to be elliptic if $A^{-1}(x, \xi)$ exists for all $(x, \xi) \in \mathbb{R}^{2n}$ and belongs to $S(g^{-1})$.
- (iii) We say that A admits an asymptotic expansion in powers of \hbar in $S(g)$ if there exists $\hbar_0 \in]0, 1]$ and a sequence of \hbar -independent symbols $(A_j = A_j(x, \xi))_{j \in \mathbb{N}} \subset S(g)$ such that A is a map from $]0, \hbar_0]$ into $S(g)$ satisfying

$$\hbar^{-(N+1)} \left(A(x, \xi; \hbar) - \sum_{j=0}^N \hbar^j A_j(x, \xi) \right) \in S(g), \quad \forall N \in \mathbb{N}. \quad (2.3)$$

If (2.3) holds, we write $A(x, \xi; \hbar) \sim \sum_{j \geq 0} \hbar^j A_j(x, \xi)$ in $S(g)$. A_0 is called the principal symbol and A_1 is called the sub-principal symbol of A .

Elements of $S(g)$ which admit an asymptotic expansion in powers of \hbar are called semiclassical symbols and the corresponding Weyl operators via formula (2.1) will be denoted $A^w(x, \hbar D_x; \hbar)$ and called \hbar -pseudodifferential operators. We denote $S_{sc}(g)$ the set of semiclassical symbols in $S(g)$.

- (iv) Let P and Q be two symbols in some suitable classes of symbols. The Moyal bracket of P, Q denoted $\{P, Q\}^*$ is defined as the Weyl symbol of the operator $i\hbar^{-1}[P^w, Q^w]$. In the following, when $\{P, Q\}^*$ admits an asymptotic expansion in powers of \hbar , the coefficient of \hbar^j will be denoted $\{P, Q\}_j^*$.

The notion of the Moyal bracket will play an important role in this paper. We refer to the appendix A for more details.

Let $H(x, \xi; \hbar) \sim \sum_{j \geq 0} \hbar^j H_j(x, \xi)$ in $S(g)$ be a $(m \times m)$ semiclassical Hamiltonian. To simplify the presentation and without any loss of generality, we suppose that $H(x, \xi; \hbar) = H_0(x, \xi) + \hbar H_1(x, \xi)$. We assume that

(A0). H_0 and H_1 are hermitian-valued and $(H_0 + i)$ is elliptic, i.e. there exists a constant $C > 0$ such that

$$\|H_0(x, \xi) + i\| \geq C g(x, \xi), \quad \forall (x, \xi) \in \mathbb{R}^{2n}.$$

Under this assumption, $H^w(x, \hbar D_x; \hbar)$ is essentially self-adjoint in $L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$ for \hbar small enough (see [12, Proposition 8.5] for the case $m = 1$). By Stone's theorem (see e.g. [24, p. 74]), the corresponding Schrödinger equation

$$i\hbar \partial_t u(t) = H^w(x, \hbar D_x; \hbar) u(t)$$

generates a one parameter group of unitary operators $U_H(t) := e^{-\frac{it}{\hbar} H^w}$ defined for all $t \in \mathbb{R}$.

Let $Q(x, \xi; \hbar) \sim \sum_{j \geq 0} \hbar^j Q_j(x, \xi)$ in $S(1)$ be a $(m \times m)$ semiclassical observable and we consider the time evolution of $Q^w(x, \hbar D_x; \hbar)$ in the Heisenberg picture given by

$$Q(t) := U_H(-t) Q^w(x, \hbar D_x; \hbar) U_H(t), \quad t \in \mathbb{R}.$$

By the Calderón-Vaillancourt theorem (Theorem A.5), $Q^w(x, \hbar D_x; \hbar)$ is bounded on $L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$ and then $Q(t)$ is uniformly bounded on $L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$ with respect to $t \in \mathbb{R}$. Moreover, $Q(t)$ satisfies the following Heisenberg equation of motion

$$\frac{d}{dt} Q(t) = \frac{i}{\hbar} [H^w, Q(t)], \quad Q(t)|_{t=0} = Q^w(x, \hbar D_x; \hbar). \quad (2.4)$$

As indicated in the introduction, the first step in the semiclassical approximation of $Q(t)$ consists in the construction of a formal asymptotic expansion in powers of \hbar for $Q(t)$ by solving the following Cauchy problems arising from (2.4) if one assumes that $Q(t)$ admits a Weyl symbol $q(t) \sim \sum_{j \geq 0} \hbar^j q_j(t)$

$$(\mathcal{C}_j) \begin{cases} \frac{d}{dt} q_j(t) &= \{H, q(t)\}_j^*, \\ q_j(t)|_{t=0} &= Q_j. \end{cases} \quad (2.5)$$

For $j = 0$, according to (1.7) it is necessary to ensure the following commutativity property

$$[H_0, q_0(t)] = 0, \quad \forall t \in \mathbb{R}. \quad (2.6)$$

For $t = 0$, since $q_0(t)|_{t=0} = Q_0$, (2.6) is equivalent to a block-diagonal form of Q_0 with respect to the eigenprojectors of H_0 . However, if one restricts to such observable, nothing ensures that this block-diagonal form will be respected by the time evolution.

2.1 Hamiltonian with scalar principal symbol

We begin with a particular but an important case where the principal symbol H_0 is a scalar multiple of the identity, that is

(A1'). $H_0(x, \xi) = \lambda(x, \xi)I_m$, for a scalar real-valued symbol λ .

This case allows us to understand the contribution of the sub-principal symbol H_1 in the time evolution. It will be clear from Theorem 2.2 below that this case is different from the scalar one studied by Bouzouina-Robert [6]. In particular, this case cannot be deduced from the results of [6].

We assume that

(A2'). For all $\gamma \in \mathbb{N}^{2n}$ and $j \in \{0, 1\}$,

$$\partial_{(x, \xi)}^\gamma H_j \in L^\infty(\mathbb{R}^{2n}), \quad \text{for } |\gamma| + j \geq 2.$$

Let ϕ_λ^t be the Hamiltonian flow generated by λ . Under the above assumption, the correspondent vector field $\mathcal{X}_\lambda := (\partial_\xi \lambda, -\partial_x \lambda)$ grows at most linearly at infinity. Therefore a trajectory $\phi_\lambda^t(x, \xi)$ cannot blow up at finite times so that, for all $(x, \xi) \in \mathbb{R}^{2n}$, $\phi_\lambda^t(x, \xi)$ exists for all $t \in \mathbb{R}$.

Put

$$\Gamma := \|J\nabla_{(x, \xi)}^{(2)} \lambda(x, \xi)\|_{L^\infty(\mathbb{R}^{2n})}, \quad (2.7)$$

where $\nabla_{(x, \xi)}^{(2)} \lambda$ is the Hessian matrix of λ and J is the $(2n \times 2n)$ matrix associated to the canonical symplectic form on \mathbb{R}^{2n} (see (A.1)).

Theorem 2.2 *Assume (A0), (A1') and (A2'), and let $Q \in S_{sc}(1)$. There exists a sequence of $(m \times m)$ matrix-valued \hbar -pseudodifferential operators $((q_j(t))^w(x, \hbar D_x))_{j \geq 0}$ such that for all $N \in \mathbb{N}$, there exists $C_N > 0$ such that for all $t \in \mathbb{R}$, the following estimate holds*

$$\left\| Q(t) - \sum_{j=0}^N \hbar^j (q_j(t))^w(x, \hbar D_x) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \leq C_N \hbar^{N+1} \exp\left((4N + \delta_n)\Gamma|t|\right), \quad (2.8)$$

where δ_n is an integer depending only on the dimension n . The symbols $q_j(t)$, $j \geq 0$, are defined by formula (3.6) and satisfy estimates (3.10) and (3.11). In particular, the principal symbol $q_0(t)$ is given by

$$q_0(t, x, \xi) = T^{-1}(t, x, \xi) Q_0(\phi_\lambda^t(x, \xi)) T(t, x, \xi), \quad t \in \mathbb{R}, (x, \xi) \in \mathbb{R}^{2n},$$

where T is the unitary $(m \times m)$ matrix-valued function solution of the system

$$\frac{d}{dt} T(t, x, \xi) = -i H_1(\phi_\lambda^t(x, \xi)) T(t, x, \xi), \quad T(0, x, \xi) = I_m. \quad (2.9)$$

As a consequence of estimate (2.8), we get the following corollary about the Ehrenfest time for the validity of the semiclassical approximation.

Corollary 2.3 *Under the assumptions of Theorem 2.2, for all $N \geq 1$, there exists $C_N > 0$ such that for every $\varepsilon > 0$, we have*

$$\left\| Q(t) - \sum_{j=0}^N \hbar^j (q_j(t))^w(x, \hbar D_x) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \leq C_N \hbar^{\varepsilon N+1} \hbar^{\frac{(\varepsilon-1)}{4} \delta_n}, \quad (2.10)$$

uniformly for $|t| \leq \frac{(1-\varepsilon)}{4\Gamma} \log(\hbar^{-1})$.

Remark 2.4 (i) The upper bound Γ is used to control the exponential growth of the flow ϕ_λ^t at infinity (see Lemma 3.4).

(ii) The constant δ_n is related to the universal constant in the Calderón-Vaillancourt Theorem (Theorem A.5). See the end of the proof of Theorem 2.2.

(iii) Notice that for $m \geq 2$, our estimate on the remainder term (2.8) is different from the one proved in the scalar case (see [6, Theorem 1.4]) where the argument in the exponential term was $2N + \delta'_n$ with δ'_n a universal constant. In particular, the constant $\frac{1}{4\Gamma}$ in the Ehrenfest time up to which the semiclassical approximation remains valid is half of the one proved in [6]. This is due to the matrix structure of the sub-principal symbol H_1 (see Remark 3.7 for more details).

2.2 General case

Now we drop the assumption (A1'). We assume that

(A1). There exists $l \in \{1, \dots, m\}$ and $r_1, \dots, r_l \in \mathbb{N}^*$ with $r_1 + \dots + r_l = m$ such that $H_0(x, \xi)$ admits exactly l distinct eigenvalues $\lambda_1(x, \xi) < \dots < \lambda_l(x, \xi)$ with constant multiplicities on \mathbb{R}^{2n} given by r_1, \dots, r_l respectively, satisfying : there exists a constant $\rho > 0$ such that for all $1 \leq \mu \neq \nu \leq l$,

$$|\lambda_\mu(x, \xi) - \lambda_\nu(x, \xi)| \geq \rho g(x, \xi), \quad \text{for } |x| + |\xi| \geq c > 0. \quad (2.11)$$

(A2). For all $\gamma \in \mathbb{N}^{2n}$ and $j \in \{0, 1\}$,

$$\partial_{(x, \xi)}^\gamma H_j \in L^\infty(\mathbb{R}^{2n}), \quad \text{for } |\gamma| + j \geq 1.$$

For $\nu \in \{1, \dots, l\}$, let $P_{\nu,0}(x, \xi)$ be the eigenprojector associated to the eigenvalue $\lambda_\nu(x, \xi)$. The assumption **(A1)** ensures that the functions $(x, \xi) \mapsto \lambda_\nu(x, \xi)$ and $(x, \xi) \mapsto P_{\nu,0}(x, \xi)$ are smooth in \mathbb{R}^{2n} . Moreover, in Lemma C.1, we show that $P_{\nu,0} \in S(1)$ and $\lambda_\nu \in S(g)$, for all $1 \leq \nu \leq l$.

As in [3] see also [19, 27] and Theorem 4.1, we construct l \hbar -pseudodifferential operators $P_1^w(x, \hbar D_x; \hbar)$, ..., $P_l^w(x, \hbar D_x; \hbar)$ satisfying

$$(P_\nu^w)^2 = (P_\nu^w)^* = P_\nu^w,$$

and

$$[H^w, P_\nu^w] = 0, \quad \sum_{\nu=1}^l P_\nu^w = \text{id}_{L^2(\mathbb{R}^n) \otimes \mathbb{C}^m}, \quad P_\nu^w P_\mu^w = 0, \quad \forall 1 \leq \nu \neq \mu \leq l,$$

modulo $\mathcal{O}(\hbar^\infty)$ in norm $\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)$. For $\nu \in \{1, \dots, l\}$, the principal symbol of $P_\nu^w(x, \hbar D_x; \hbar)$ coincides with the eigenprojector $P_{\nu,0}$. The operators $(P_\nu^w(x, \hbar D_x; \hbar))_{1 \leq \nu \leq l}$ are called semiclassical projections associated to $H^w(x, \hbar D_x; \hbar)$.

As indicated in [3, Proposition 3.2] (see also Remark 2.8), to construct a complete asymptotic expansion in powers of \hbar for $Q(t)$, some restrictions on the initial observable Q are necessary. We introduce the class $\mathcal{Q}(1)$ of observables $Q \in S_{\text{sc}}(1)$ that are "semiclassically" block-diagonal with respect to the semiclassical projections P_ν , $1 \leq \nu \leq l$, i.e.

$$\mathcal{Q}(1) := \left\{ Q \in S_{\text{sc}}(1); Q \sim \sum_{\nu=1}^l P_\nu \# Q \# P_\nu \text{ in } S(1) \right\}.$$

In particular, using formula (A.8), one sees that if $Q \in \mathcal{Q}(1)$ then Q_0 is block diagonal with respect to the eigenprojectors $P_{\nu,0}$, i.e.

$$Q_0(x, \xi) = \sum_{\nu=1}^l P_{\nu,0}(x, \xi) Q_0(x, \xi) P_{\nu,0}(x, \xi), \quad \forall (x, \xi) \in \mathbb{R}^{2n}.$$

Let ϕ_ν^t be the Hamiltonian flow generated by the eigenvalue λ_ν . The assumption **(A2)** ensures that $\phi_\nu^t(x, \xi)$ exists globally on \mathbb{R} , for all $(x, \xi) \in \mathbb{R}^{2n}$, $1 \leq \nu \leq l$.

Put

$$\Gamma_v := \|J\nabla_{(x,\xi)}^{(2)} \lambda_v(x, \xi)\|_{L^\infty(\mathbb{R}^{2n})}, \quad \Gamma_{\max} := \max_{1 \leq v \leq l} \Gamma_v, \quad (2.12)$$

where $\nabla_{(x,\xi)}^{(2)} \lambda_v$ denotes the Hessian matrix of λ_v , $1 \leq v \leq l$.

Our main result of this paper is the following

Theorem 2.5 *Assume (A0-2) and let $Q \in \mathcal{Q}(1)$. There exists a sequence $((q_j(t))^w(x, \hbar D_x))_{j \geq 0}$ of $(m \times m)$ matrix-valued \hbar -pseudodifferential operators such that for all $N \in \mathbb{N}$, there exists $C_N > 0$ such that for all $t \in \mathbb{R}$, the following estimate holds*

$$\left\| Q(t) - \sum_{j=0}^N \hbar^j (q_j(t))^w(x, \hbar D_x) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \leq C_N \hbar^{N+1} \exp\left((4N + \tilde{\delta}_n) \Gamma_{\max} |t|\right), \quad (2.13)$$

where $\tilde{\delta}_n$ is an integer depending only on the dimension n . The symbols $q_j(t, x, \xi)$ are defined for $t \in \mathbb{R}$ and $(x, \xi) \in \mathbb{R}^{2n}$ by

$$q_j(t, x, \xi) := \sum_{v=1}^l q_{v,j}(t, x, \xi), \quad j \geq 0,$$

where $q_{v,j}(t)$ are given by the general formula (4.29) and satisfy estimates (4.53) and (4.54). In particular, the principal symbol $q_0(t)$ is given by

$$q_0(t, x, \xi) = \sum_{v=1}^l T_v^{-1}(t, x, \xi) (P_{v,0} Q_0 P_{v,0}) (\phi_v^t(x, \xi)) T_v(t, x, \xi), \quad (2.14)$$

where T_v is the unitary $(m \times m)$ matrix-valued function solution of the system

$$\frac{d}{dt} T_v(t, x, \xi) = -i \tilde{H}_{v,1}(\phi_v^t(x, \xi)) T_v(t, x, \xi) \quad T_v(0, x, \xi) = I_m. \quad (2.15)$$

Here $\tilde{H}_{v,1}$ is the $(m \times m)$ hermitian-valued function defined by

$$\tilde{H}_{v,1} = \frac{1}{2i} P_{v,0} \{P_{v,0}, H_0\} P_{v,0} - i [P_{v,0}, \{\lambda_v, P_{v,0}\}] + P_{v,0} H_1 P_{v,0}. \quad (2.16)$$

As a consequence we get the following corollary.

Corollary 2.6 *Under the assumptions of Theorem 2.5, for all $N \geq 1$ there exists $C_N > 0$ such that for all $\varepsilon > 0$, we have*

$$\left\| Q(t) - \sum_{j=0}^N \hbar^j (q_j(t))^w(x, \hbar D_x) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \leq C_N \hbar^{\varepsilon N+1} \hbar^{\frac{(\varepsilon-1)}{4} \tilde{\delta}_n}, \quad (2.17)$$

uniformly for $|t| \leq \frac{(1-\varepsilon)}{4\Gamma_{\max}} \log(\hbar^{-1})$.

If we only look for the principal symbol of $Q(t)$, the assumption $Q \in \mathcal{Q}(1)$ can be relaxed and we have the following result.

Corollary 2.7 *Let H be a semiclassical Hamiltonian satisfying the assumptions of Theorem 2.5 and let $Q \in S_{sc}(1)$. We assume that $Q_0(x, \xi) = \sum_{v=1}^l P_{v,0}(x, \xi) \tilde{Q}(x, \xi) P_{v,0}(x, \xi)$ for some $\tilde{Q} \in S(1)$. There exists $C > 0$ such that for all $t \in \mathbb{R}$, the following estimate holds*

$$\left\| Q(t) - (q_0(t))^w(x, \hbar D_x) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \leq C \hbar \exp(\tilde{\delta}_n \Gamma_{\max} |t|),$$

where $q_0(t)$ is given by (2.14).

Remark 2.8 *In [3, Proposition 3.4], it was shown that the class $\mathcal{Q}(1)$ exhausts all symbols $Q \in S_{sc}(1)$ such that the corresponding Heisenberg observable $Q(t)$ is an \hbar -pseudodifferential operator with symbol $q(t) \in S_{sc}(1)$, for all finite time t . More explicitly, if H satisfies the assumptions of Theorem 2.5, then we have*

$$\left\{ Q \in S_{sc}(1); \forall |t| \leq \bar{t} < \infty, Q(t) = (q(t))^w(x, \hbar D_x; \hbar), \text{ with } q(t) \in S_{sc}(1) \right\} = \mathcal{Q}(1).$$

3 Hamiltonian with scalar principal symbol

In this section, we study the particular case where the principal symbol H_0 is a scalar multiple of the identity in $M_m(\mathbb{C})$. The proof of Theorem 2.2 relies essentially on the following steps. In the next paragraph, using assumption **(A1')**, we construct a formal asymptotic expansion in powers of \hbar for $Q(t)$ by solving the Cauchy problems $(\mathcal{C}_j)_{j \geq 0}$ (see (2.5)). The constructed matrix-valued functions $(q_j(t, x, \xi))_{j \geq 0}$ are defined by formula (3.6). Since we are interested in the semiclassical approximation for $Q(t)$ up to times of Ehrenfest type, we give in Proposition 3.2 uniform (in time) estimates on the derivatives with respect to (x, ξ) of the symbols $(q_j(t, x, \xi))_{j \geq 0}$. Then, using these estimates, we prove (2.8) by following the method of Bouzouina-Robert [6].

3.1 Formal asymptotic expansion

Let $H(x, \xi; \hbar) = H_0(x, \xi) + \hbar H_1(x, \xi)$ be a semiclassical Hamiltonian and suppose that H_0 satisfies **(A1')**. According to this assumption, the principal symbol of $[H, q(t)]_\#$ given by the commutator $[H_0, q_0(t)]$ vanishes for all $t \in \mathbb{R}$. Consequently, using the rule of asymptotic expansion of the product of symbols (formula (A.5)), the symbol $\{H, q(t)\}^*$ can be expended in a power series of \hbar (see formula (A.9)) and then the Cauchy problems $(\mathcal{C}_j)_{j \geq 0}$ become

$$(\mathcal{C}_j) \begin{cases} \frac{d}{dt} q_j(t) &= \sum_{|\alpha|+|\beta|+k+p=j+1} \tilde{\gamma}(\alpha, \beta) \left(H_{k(\alpha)}^{(\beta)} q_p(t)_{(\beta)}^{(\alpha)} - (-1)^{|\alpha|-|\beta|} q_p(t)_{(\beta)}^{(\alpha)} H_{k(\alpha)}^{(\beta)} \right) \\ q_j(t)|_{t=0} &= Q_j, \end{cases} \quad (3.1)$$

$$\text{with } \tilde{\gamma}(\alpha, \beta) := \frac{i(-1)^{|\beta|}}{(2i)^{|\alpha|+|\beta|} \alpha! \beta!}.$$

Thanks to assumption **(A1')** again, for $p = j + 1$, the right hand side of (3.1) is equal to $i[H_0, q_{j+1}(t)]$ which vanishes for all $t \in \mathbb{R}$. Then, (\mathcal{C}_j) can be rewritten in the following form

$$\begin{aligned} \frac{d}{dt} q_j(t) &= \sum_{\substack{|\alpha|+|\beta|+k+p=j+1 \\ 0 \leq p \leq j}} \tilde{\gamma}(\alpha, \beta) \left(H_{k(\alpha)}^{(\beta)} q_p(t)_{(\beta)}^{(\alpha)} - (-1)^{|\alpha|-|\beta|} q_p(t)_{(\beta)}^{(\alpha)} H_{k(\alpha)}^{(\beta)} \right) \\ &= \{\lambda, q_j(t)\} + i[H_1, q_j(t)] + \sum_{\substack{|\alpha|+|\beta|+k+p=j+1 \\ 0 \leq p \leq j-1}} \tilde{\gamma}(\alpha, \beta) \left(H_{k(\alpha)}^{(\beta)} q_p(t)_{(\beta)}^{(\alpha)} - (-1)^{|\alpha|-|\beta|} q_p(t)_{(\beta)}^{(\alpha)} H_{k(\alpha)}^{(\beta)} \right). \end{aligned} \quad (3.2)$$

For $j \geq 0$, we set

$$B_j(t, x, \xi) := \sum_{\substack{|\alpha|+|\beta|+k+p=j+1 \\ 0 \leq p \leq j-1}} \tilde{\gamma}(\alpha, \beta) \left(H_{k(\alpha)}^{(\beta)} q_p(t)_{(\beta)}^{(\alpha)} - (-1)^{|\alpha|-|\beta|} q_p(t)_{(\beta)}^{(\alpha)} H_{k(\alpha)}^{(\beta)} \right)(x, \xi), \quad (3.3)$$

with the convention $B_0 = 0$ since the sum is empty. Before giving the solution of (3.2), let us make the following remark concerning the case where the sub-principal symbol H_1 is also scalar-valued.

Remark 3.1 Suppose that H_1 is a scalar real-valued symbol. Then, $[H_1, q_j(t)]$ vanishes and one can easily verify that equation (3.2) is equivalent to the following one

$$\frac{d}{dt} \left(q_j(t, \phi_\lambda^{-t}(x, \xi)) \right) = B_j(t, \phi_\lambda^{-t}(x, \xi)), \quad j \geq 0,$$

where B_j can be rewritten in the simpler form

$$B_j(t, x, \xi) = \sum_{\substack{|\alpha|+|\beta|+k+p=j+1 \\ 0 \leq p \leq j-1}} \frac{i((-1)^{|\beta|} - (-1)^{|\alpha|})}{(2i)^{|\alpha|+|\beta|} \alpha! \beta!} \left(H_{k(\alpha)}^{(\beta)}(x, \xi) q_p(t)_{(\beta)}^{(\alpha)}(x, \xi) \right). \quad (3.4)$$

Consequently, the solutions $q_{j,sca}(t)$, $j \geq 0$, are given by

$$q_{j,sca}(t, x, \xi) = q_{j,sca}(0, \phi_\lambda^t(x, \xi)) + \int_0^t B_j(s, \phi_\lambda^{t-s}(x, \xi)) ds, \quad t \in \mathbb{R}, (x, \xi) \in \mathbb{R}^{2n}, \quad (3.5)$$

where we introduced the index "sca" to precise that we are in the case where H_0 and H_1 are scalar-valued. In particular,

$$q_{0,sca}(t, x, \xi) = Q_0 \circ \phi_\lambda^t(x, \xi) \quad \text{and} \quad q_{1,sca}(t, x, \xi) = Q_1(\phi_\lambda^t(x, \xi)) + \int_0^t \{H_1, Q_0(\phi_\lambda^s)\} \circ \phi_\lambda^{t-s}(x, \xi) ds.$$

□

Turn now to the resolution of (3.2). Applying the results of Appendix B with $\Lambda = \lambda$, $A = H_1$ and $B(t) = B_j(t)$, we get the solution for all $j \geq 0$

$$q_j(t, x, \xi) = T^{-1}(t, x, \xi) \left(Q_j(\phi_\lambda^t(x, \xi)) + \int_0^t T^{-1}(-s, \phi_\lambda^t(x, \xi)) B_j(s, \phi_\lambda^{t-s}(x, \xi)) T(-s, \phi_\lambda^t(x, \xi)) ds \right) T(t, x, \xi), \quad (3.6)$$

defined for all $t \in \mathbb{R}$ and $(x, \xi) \in \mathbb{R}^{2n}$, where T and T^{-1} are the unitary $(m \times m)$ matrix-valued functions solutions of the following systems

$$\frac{d}{dt} T(t, x, \xi) = -i H_1(\phi_\lambda^t(x, \xi)) T(t, x, \xi), \quad T(0, x, \xi) = I_m, \quad (3.7)$$

$$\frac{d}{dt} T^{-1}(t, x, \xi) = i T^{-1}(t, x, \xi) H_1(\phi_\lambda^t(x, \xi)), \quad T^{-1}(0, x, \xi) = I_m. \quad (3.8)$$

In particular, the principal symbol $q_0(t)$ is given by

$$q_0(t, x, \xi) = T^{-1}(t, x, \xi) Q_0(\phi_\lambda^t(x, \xi)) T(t, x, \xi), \quad \forall t \in \mathbb{R}, (x, \xi) \in \mathbb{R}^{2n}. \quad (3.9)$$

3.2 Uniform estimates

Let Γ be the upper bound defined by (2.7).

Proposition 3.2 *Assume (A2'). For all $\gamma \in \mathbb{N}^{2n}$ and all $j \geq 0$, there exists $C_{\gamma,j} > 0$ such that for all $t \in \mathbb{R}$ and all $(x, \xi) \in \mathbb{R}^{2n}$, we have*

$$\|\partial_{(x,\xi)}^\gamma q_0(t, x, \xi)\| \leq C_{\gamma,0} \exp(|\gamma| \Gamma |t|), \quad (3.10)$$

and for $j \geq 1$,

$$\|\partial_{(x,\xi)}^\gamma q_j(t, x, \xi)\| \leq C_{\gamma,j} \exp((2|\gamma| + 4j - 3) \Gamma |t|). \quad (3.11)$$

To prove this proposition we need to recall the multivariate Faá Di Bruno formula used for computing arbitrary partial derivatives of a function composition. In the following, this formula will be used wherever we have to estimate the derivatives of observables moving along the Hamiltonian flow. In the literature, one can found several forms to this formula (see for instance [22], [10]). As in [6], we use the following one :

Lemma 3.3 *Let $F = (F_{ij})_{1 \leq i,j \leq m} : \mathbb{R}^{2n} \rightarrow M_m(\mathbb{C})$ and $G = (G_1, \dots, G_{2n}) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be smooth functions. For all $\gamma \in \mathbb{N}^{2n}$, we have*

$$\partial_{(x,\xi)}^\gamma (F \circ G) = \left(\partial_{(x,\xi)}^\gamma (F_{ij} \circ G) \right)_{1 \leq i,j \leq m}$$

where

$$\partial_{(x,\xi)}^\gamma (F_{ij} \circ G) = \sum_{\substack{\beta \in \mathbb{N}^{2n} \\ 0 \neq \beta \leq \gamma}} (\partial_{(x,\xi)}^\beta F_{ij}) \circ G \cdot \mathcal{A}_{\gamma,\beta}(G), \quad (3.12)$$

with

$$\mathcal{A}_{\gamma,\beta}(G) = \gamma! \sum_{\substack{\alpha=\beta \\ \sum \alpha|\eta|=\gamma}} \prod_{\alpha \in \mathbb{N}^{2n} \setminus \{0\}} \frac{1}{\eta!} \left(\frac{\partial_{(x,\xi)}^\alpha G_1}{\alpha!} \right)^{\eta_1} \dots \left(\frac{\partial_{(x,\xi)}^\alpha G_{2n}}{\alpha!} \right)^{\eta_{2n}}. \quad (3.13)$$

Here we use the usual rules for multi-indices (see [10]).

The proof of Proposition 3.2 is based on the three following lemmas. The first one gives exponential estimate on the derivatives (with respect to (x, ξ)) of the Hamiltonian flow associated to λ . This result can be proved by induction on $|\gamma|$ using the Jacobi stability equation

$$\frac{d}{dt} \nabla_{(x,\xi)} \phi_\lambda^t(x, \xi) = J \nabla_{(x,\xi)}^{(2)} \lambda(\phi_\lambda^t(x, \xi)) \nabla_{(x,\xi)} \phi_\lambda^t(x, \xi), \quad (3.14)$$

where $\nabla_{(x,\xi)} \phi_\lambda^t := (\partial_x \phi_\lambda^t, \partial_\xi \phi_\lambda^t)$.

The following lemma is proved in [6, Lemma 2.2].

Lemma 3.4 *We assume that*

$$\partial_{(x,\xi)}^\gamma \lambda \in L^\infty(\mathbb{R}^{2n}), \quad \forall \gamma \in \mathbb{N}^{2n}; |\gamma| \geq 2.$$

Then, for all $\gamma \in \mathbb{N}^{2n} \setminus \{0\}$, there exists $C_\gamma > 0$ such that for all $t \in \mathbb{R}$ and all $(x, \xi) \in \mathbb{R}^{2n}$,

$$\|\partial_{(x,\xi)}^\gamma \phi_\lambda^t(x, \xi)\| \leq C_\gamma \exp(|\gamma| \Gamma |t|). \quad (3.15)$$

In the next lemma, we prove similar estimate on the derivatives of the matrix-valued function T (see (3.7)).

Lemma 3.5 *Assume (A2'). For all $\gamma \in \mathbb{N}^{2n} \setminus \{0\}$, there exists $C_\gamma > 0$ (independent of $t \in \mathbb{R}$ and $(x, \xi) \in \mathbb{R}^{2n}$) such that*

$$\|\partial_{(x,\xi)}^\gamma T(t, x, \xi)\| \leq C_\gamma \exp(|\gamma| \Gamma |t|). \quad (3.16)$$

Proof. Without any loss of generality, we assume that $t \geq 0$ (the proof for $t \leq 0$ is similar). We proceed by induction on $|\gamma|$. Let us check (3.16) for the first order derivative of T with respect to x_1 . A straightforward computation using equations (3.7) and (3.8) yields

$$\begin{aligned} \frac{d}{dt} (T^{-1}(t, x, \xi) \partial_{x_1} T(t, x, \xi)) &= \partial_t T^{-1}(t, x, \xi) \partial_{x_1} T(t, x, \xi) + T^{-1}(t, x, \xi) \partial_t \partial_{x_1} T(t, x, \xi) \\ &= -i T^{-1}(t, x, \xi) (\partial_{x_1} H_1) (\phi_\lambda^t(x, \xi)) \partial_{x_1} \phi_\lambda^t(x, \xi) T(t, x, \xi). \end{aligned}$$

Therefore

$$T^{-1}(t, x, \xi) \partial_{x_1} T(t, x, \xi) = -i \int_0^t T^{-1}(s, x, \xi) (\partial_{x_1} H_1) (\phi_\lambda^s(x, \xi)) \partial_{x_1} \phi_\lambda^s(x, \xi) T(s, x, \xi) ds,$$

since $\partial_{x_1} T(0, x, \xi) = 0$ (we recall that $T(0, x, \xi) = I_m$). Taking into account the fact that T^{-1} is unitary and using estimate (3.15), we obtain

$$\|\partial_{x_1} T(t, x, \xi)\| \leq \int_0^t \|\partial_{x_1} \phi_\lambda^s(x, \xi)\| \cdot \|\partial_{x_1} H_1\|_{L^\infty(\mathbb{R}^{2n})} ds \leq C \exp(\Gamma t),$$

uniformly for $t \geq 0$ and $(x, \xi) \in \mathbb{R}^{2n}$. This gives the proof for $\gamma = (1, 0, \dots, 0)$. The same proof holds for $|\gamma| = 1$.

Let us now assume that (3.16) holds for all $\gamma \in \mathbb{N}^{2n}$ with $|\gamma| < r$, $r \geq 2$, and take $|\gamma| = r$. Computing derivatives with respect to (x, ξ) in (3.7) using Leibniz formula, we get

$$\frac{d}{dt} \partial_{(x,\xi)}^\gamma T(t, x, \xi) = -i H_1(\phi_\lambda^t(x, \xi)) \partial_{(x,\xi)}^\gamma T(t, x, \xi) - i \sum_{1 \leq |\beta| \leq r} \binom{\beta}{\gamma} \partial_{(x,\xi)}^\beta (H_1(\phi_\lambda^t(x, \xi))) \partial_{(x,\xi)}^{\gamma-\beta} T(t, x, \xi).$$

Therefore

$$\frac{d}{dt}(T^{-1}(t, x, \xi) \partial_{(x, \xi)}^\gamma T(t, x, \xi)) = -i T^{-1}(t, x, \xi) \sum_{1 \leq |\beta| \leq r} \binom{\beta}{\gamma} \partial_{(x, \xi)}^\beta (H_1(\phi_\lambda^t(x, \xi))) \partial_{(x, \xi)}^{\gamma-\beta} T(t, x, \xi).$$

According to assumption **(A2')**, for all $\beta \in \mathbb{N}^{2n}$ with $|\beta| \geq 1$, we have $\partial_{(x, \xi)}^\beta H_1 \in L^\infty(\mathbb{R}^{2n})$. Consequently, using Faà Di Bruno's formula (3.12) and estimate (3.15), we obtain

$$\|\partial_{(x, \xi)}^\beta (H_1 \circ \phi_\lambda^t(x, \xi))\| \leq C_\beta \exp(|\beta| \Gamma t), \quad (3.17)$$

uniformly with respect to $t \geq 0$ and $(x, \xi) \in \mathbb{R}^{2n}$.

On the other hand, by the induction hypothesis, there exists $C_{\gamma, \beta} > 0$ such that for all $t \geq 0$ and all $(x, \xi) \in \mathbb{R}^{2n}$, we have

$$\|\partial_{(x, \xi)}^{\gamma-\beta} T(t, x, \xi)\| \leq C_{\gamma, \beta} \exp((r - |\beta|) \Gamma t). \quad (3.18)$$

Putting together (3.17) and (3.18) and taking into account the fact that $\partial_{(x, \xi)}^\gamma T(0, x, \xi) = 0$, we get

$$\begin{aligned} \|\partial_{(x, \xi)}^\gamma T(t, x, \xi)\| &\leq \sum_{1 \leq |\beta| \leq r} C_{\gamma, \beta} \int_0^t \|\partial_{(x, \xi)}^\beta (H_1(\phi_\lambda^s(x, \xi)))\| \cdot \|\partial_{(x, \xi)}^{\gamma-\beta} T(s, x, \xi)\| ds \\ &\leq \sum_{1 \leq |\beta| \leq r} C'_{\gamma, \beta} \int_0^t \exp(|\beta| \Gamma s) \exp((r - |\beta|) \Gamma s) ds \\ &\leq C_\gamma \exp(r \Gamma t). \end{aligned}$$

Hence (3.16) holds for $|\gamma| = r$. This ends the proof.

Notice that the same proof can be repeated for T^{-1} and then estimate (3.16) remains valid for the derivatives of T^{-1} . \square

The following lemma is a consequence of the two previous lemmas and the Faà Di Bruno formula (3.12).

Lemma 3.6 *Under assumption **(A2')**, for all $\gamma \in \mathbb{N}^{2n} \setminus \{0\}$, there exists $C_\gamma > 0$ such that for all $(x, \xi) \in \mathbb{R}^{2n}$ and all $t, s \in \mathbb{R}$, we have*

$$\|\partial_{(x, \xi)}^\gamma (T(s, \phi_\lambda^t(x, \xi)))\| \leq C_\gamma \exp(|\gamma| \Gamma (|t| + |s|)). \quad (3.19)$$

Furthermore, the same estimate holds for the derivatives of $T^{-1}(s, \phi_\lambda^t(x, \xi))$.

With Lemmas 3.4, 3.5 and 3.6 at hand, we are now ready to prove Proposition 3.2.

Proof.

- (i) We start by proving estimate (3.10). Using formula (3.12) and estimate (3.15), one can easily verify that for all $\gamma \in \mathbb{N}^{2n}$, there exists $C_\gamma > 0$ such that for all $t \in \mathbb{R}$ and all $(x, \xi) \in \mathbb{R}^{2n}$,

$$\|\partial_{(x, \xi)}^\gamma (Q_0 \circ \phi_\lambda^t(x, \xi))\| \leq C_\gamma \exp(|\gamma| \Gamma |t|). \quad (3.20)$$

Consequently, by differentiating $q_0(t)$ $|\gamma|$ -times with respect to (x, ξ) using the Leibniz formula, we obtain

$$\begin{aligned} \|\partial_{(x, \xi)}^\gamma q_0(t, x, \xi)\| &\leq \sum_{\beta \leq \gamma, \alpha \leq \beta} \binom{\beta}{\gamma} \binom{\alpha}{\beta} \|\partial_{(x, \xi)}^\alpha T^{-1}(t, x, \xi)\| \|\partial_{(x, \xi)}^{\beta-\alpha} (Q_0(\phi_\lambda^t(x, \xi)))\| \|\partial_{(x, \xi)}^{\gamma-\beta} T(t, x, \xi)\| \\ &\leq \sum_{\beta \leq \gamma, \alpha \leq \beta} C_{\alpha, \beta, \gamma} \exp((|\gamma| + |\alpha| - |\beta|) \Gamma |t|) \exp((|\beta| - |\alpha|) \Gamma |t|) \\ &\leq C_\gamma \exp(|\gamma| \Gamma |t|), \end{aligned}$$

uniformly for $(t, x, \xi) \in \mathbb{R} \times \mathbb{R}^{2n}$. Hence (3.10) holds.

- (ii) We shall prove (3.11) by induction with respect to $j \geq 1$. We give the proof only for $t \geq 0$, the case $t \leq 0$ is similar. Recall the expression of $q_j(t, x, \xi)$

$$q_j(t, x, \xi) = T^{-1}(t, x, \xi) \left(Q_j(\phi_\lambda^t(x, \xi)) + \int_0^t T^{-1}(-s, \phi_\lambda^t(x, \xi)) B_j(s, \phi_\lambda^{t-s}(x, \xi)) T(-s, \phi_\lambda^t(x, \xi)) ds \right) T(t, x, \xi),$$

with

$$B_j(t, x, \xi) := \sum_{\substack{|\alpha|+|\beta|+k+p=j+1 \\ 0 \leq p \leq j-1}} \tilde{\gamma}(\alpha, \beta) \left(H_{k(\alpha)}^{(\beta)} q_p(t)_{(\beta)}^{(\alpha)} - (-1)^{|\alpha|-|\beta|} q_p(t)_{(\beta)}^{(\alpha)} H_{k(\alpha)}^{(\beta)} \right) (x, \xi).$$

For $j = 1$, we have

$$B_1(t, x, \xi) = \sum_{|\alpha|+|\beta|+k=2} \tilde{\gamma}(\alpha, \beta) \left(H_{k(\alpha)}^{(\beta)} q_0(t)_{(\beta)}^{(\alpha)} - (-1)^{|\alpha|-|\beta|} q_0(t)_{(\beta)}^{(\alpha)} H_{k(\alpha)}^{(\beta)} \right) (x, \xi).$$

Since H_0 is scalar according to assumption **(A1')**, for $k = 0$ the previous sum vanishes and then B_1 can be rewritten as

$$\begin{aligned} B_1(s, x, \xi) &= \sum_{|\alpha|+|\beta|=1} \tilde{\gamma}(\alpha, \beta) \left(H_{1(\alpha)}^{(\beta)} q_0(s)_{(\beta)}^{(\alpha)} - (-1)^{|\alpha|-|\beta|} q_0(s)_{(\beta)}^{(\alpha)} H_{1(\alpha)}^{(\beta)} \right) (x, \xi) \\ &= \frac{1}{2} \left(\{H_1, q_0(s)\}(x, \xi) - \{q_0(s), H_1\}(x, \xi) \right). \end{aligned} \quad (3.21)$$

Let $\gamma \in \mathbb{N}^{2n}$. Using assumption **(A2')** with $j = 1$ and estimate (3.10), we get

$$\|\partial_{(x,\xi)}^\gamma B_1(s, x, \xi)\| \leq C_\gamma \exp\left((|\gamma| + 1)\Gamma s\right),$$

uniformly for $s \geq 0$ and $(x, \xi) \in \mathbb{R}^{2n}$. Now, computing the derivatives of $B_1 \circ \phi_\lambda^{t-s}$ by means of the Faà Di Bruno formula (3.12) and combining the above estimate with (3.15), we obtain

$$\|\partial_{(x,\xi)}^\gamma B_1(s, \phi_\lambda^{t-s}(x, \xi))\| \leq C_\gamma \exp\left((|\gamma|t + s)\Gamma\right), \quad (3.22)$$

uniformly for $0 \leq s \leq t$ and $(x, \xi) \in \mathbb{R}^{2n}$.

Put

$$A_1(t, s, x, \xi) := T^{-1}(-s, \phi_\lambda^t(x, \xi)) B_1(s, \phi_\lambda^{t-s}(x, \xi)) T(-s, \phi_\lambda^t(x, \xi)),$$

and

$$\tilde{A}_1(t, x, \xi) := Q_1(\phi_\lambda^t(x, \xi)) + \int_0^t A_1(t, s, x, \xi) ds.$$

Using Leibniz formula, estimates (3.22) and (3.19) imply

$$\begin{aligned} \|\partial_{(x,\xi)}^\gamma A_1(t, s, x, \xi)\| &\leq \sum_{\beta \leq \gamma, \alpha \leq \beta} \binom{\beta}{\gamma} \binom{\alpha}{\beta} \|\partial_{(x,\xi)}^\alpha \left(T^{-1}(-s, \phi_\lambda^t(x, \xi)) \right)\| \\ &\quad \times \|\partial_{(x,\xi)}^{\gamma-\beta} \left(T(-s, \phi_\lambda^t(x, \xi)) \right)\| \|\partial_{(x,\xi)}^{\beta-\alpha} \left(B_1(s, \phi_\lambda^{t-s}(x, \xi)) \right)\| \\ &\leq \sum_{\beta \leq \gamma, \alpha \leq \beta} C_{\alpha, \beta, \gamma} \exp\left((|\gamma| + |\alpha| - |\beta|)(t + s)\Gamma\right) \exp\left((|\beta| - |\alpha|)t + s\Gamma\right) \\ &\leq C_\gamma \exp\left((|\gamma|t + (|\gamma| + 1)s)\Gamma\right), \end{aligned} \quad (3.23)$$

uniformly for $0 \leq s \leq t$ and $(x, \xi) \in \mathbb{R}^{2n}$. Therefore,

$$\left\| \int_0^t \partial_{(x,\xi)}^\gamma A_1(t, s, x, \xi) ds \right\| \leq C_\gamma \exp(\Gamma|\gamma|t) \int_0^t \exp(\Gamma(|\gamma| + 1)s) ds \leq C'_\gamma \exp((2|\gamma| + 1)\Gamma t). \quad (3.24)$$

Combining this estimate with the fact that $Q_1 \circ \phi_\lambda^t$ satisfies estimate (3.20), we get

$$\|\partial_{(x,\xi)}^\gamma \tilde{A}_1(t, x, \xi)\| \leq C_\gamma \exp((2|\gamma| + 1)\Gamma t).$$

Finally, we use Leibniz formula again to compute derivatives with respect to (x, ξ) of $q_1(t, x, \xi)$. The above estimate together with estimate (3.16) give

$$\begin{aligned} \|\partial_{(x,\xi)}^\gamma q_1(t, x, \xi)\| &\leq \sum_{\beta \leq \gamma, \alpha \leq \beta} \binom{\beta}{\gamma} \binom{\alpha}{\beta} \|\partial_{(x,\xi)}^\alpha T^{-1}(t, x, \xi)\| \|\partial_{(x,\xi)}^{\gamma-\beta} T(t, x, \xi)\| \|\partial_{(x,\xi)}^{\beta-\alpha} \tilde{A}_1(t, x, \xi)\| \\ &\leq C_\gamma \exp((2|\gamma| + 1)\Gamma t), \end{aligned}$$

uniformly for $t \geq 0$ and $(x, \xi) \in \mathbb{R}^{2n}$. Thus we proved (3.11) for $j = 1$.

Now, suppose that (3.11) holds for all $j < r$. For $\gamma \in \mathbb{N}^{2n}$, we have

$$\partial_{(x,\xi)}^\gamma B_r(s, x, \xi) = \sum_{\substack{|\alpha|+|\beta|+k+p=r+1 \\ 0 \leq p \leq r-1}} \tilde{\gamma}(\alpha, \beta) \partial_{(x,\xi)}^\gamma \left(H_{k(\alpha)}^{(\beta)} q_p(s)_{(\beta)}^{(\alpha)} - (-1)^{|\alpha|-|\beta|} q_p(s)_{(\beta)}^{(\alpha)} H_{k(\alpha)}^{(\beta)} \right) (x, \xi). \quad (3.25)$$

We shall only focus on the first term of the above difference since the other term can be estimated similarly. Applying Leibniz formula, we get

$$\partial_{(x,\xi)}^\gamma \left(H_{k(\alpha)}^{(\beta)} q_p(s)_{(\beta)}^{(\alpha)} \right) (x, \xi) = \sum_{\eta \leq \gamma} \binom{\eta}{\gamma} \partial_{(x,\xi)}^\eta H_{k(\alpha)}^{(\beta)}(x, \xi) \partial_{(x,\xi)}^{\gamma-\eta} q_p(s)_{(\beta)}^{(\alpha)}(x, \xi).$$

Firstly, since the sum in (3.25) is over $((\alpha, \beta), k) \in \mathbb{N}^{2n} \times \{0, 1\}$ such that $|\alpha| + |\beta| + k \geq 2$, then by assumption (A2') we have $\partial_{(x,\xi)}^\eta H_{k(\alpha)}^{(\beta)} \in L^\infty(\mathbb{R}^{2n})$, for all $\eta \in \mathbb{N}^{2n}$.

On the other hand, by the induction hypothesis, there exists a constant $C = C(\gamma, \eta, \alpha, \beta) > 0$ such that for all $s \geq 0$ and $(x, \xi) \in \mathbb{R}^{2n}$, we have

$$\|\partial_{(x,\xi)}^{\gamma-\eta} q_p(s)_{(\beta)}^{(\alpha)}(x, \xi)\| \leq C \exp\left((2(|\gamma| - |\eta|) + |\alpha| + |\beta|) + 4p - 3\right)\Gamma s. \quad (3.26)$$

Thus, taking the supremum over $0 \leq |\eta| \leq |\gamma|$ and $|\alpha| + |\beta| = r + 1 - p$ with $0 \leq p \leq r - 1$, we obtain

$$\|\partial_{(x,\xi)}^\gamma B_r(s, x, \xi)\| \leq C_\gamma \exp((2|\gamma| + 4r - 3)\Gamma s),$$

uniformly with respect to $s \geq 0$ and $(x, \xi) \in \mathbb{R}^{2n}$.

Consequently, by applying Faá Di Bruno's formula (3.12) and using estimate the flow (3.15), we get

$$\|\partial_{(x,\xi)}^\gamma B_r(s, \phi_\lambda^{t-s}(x, \xi))\| \leq C_\gamma \exp\left((2|\gamma| + 4r - 3)\Gamma s + |\gamma|\Gamma(t - s)\right), \quad (3.27)$$

uniformly for $0 \leq s \leq t$ and $(x, \xi) \in \mathbb{R}^{2n}$. Put

$$A_r(t, s, x, \xi) := T^{-1}(-s, \phi_\lambda^t(x, \xi)) B_r(s, \phi_\lambda^{t-s}(x, \xi)) T(-s, \phi_\lambda^t(x, \xi)),$$

and

$$\tilde{A}_r(t, x, \xi) := Q_r(\phi_\lambda^t(x, \xi)) + \int_0^t A_r(t, s, x, \xi) ds.$$

Performing a similar computation as for A_1 and using estimates (3.27) and (3.6), we obtain

$$\left\| \int_0^t \partial_{(x,\xi)}^\gamma A_r(t, s, x, \xi) ds \right\| \leq C_\gamma \exp\left((2|\gamma| + 4r - 3)\Gamma t\right),$$

uniformly for $t \geq 0$ and $(x, \xi) \in \mathbb{R}^{2n}$. Consequently, using the fact that $Q_r \circ \phi_\lambda^t$ satisfies the estimate (3.20), we get

$$\|\partial_{(x,\xi)}^\gamma \tilde{A}_r(t, x, \xi)\| \leq C_\gamma \exp\left((2|\gamma| + 4r - 3)\Gamma t\right).$$

Finally, using the Leibniz formula and (3.16), we conclude

$$\begin{aligned} \|\partial_{(x,\xi)}^\gamma q_r(t, x, \xi)\| &\leq \sum_{\beta \leq \gamma, \alpha \leq \beta} \binom{\beta}{\gamma} \binom{\alpha}{\beta} \|\partial_{(x,\xi)}^\alpha T^{-1}(t, x, \xi)\| \|\partial_{(x,\xi)}^{\gamma-\beta} T(t, x, \xi)\| \|\partial_{(x,\xi)}^{\beta-\alpha} \tilde{A}_r(t, x, \xi)\| \\ &\leq C_\gamma \exp\left((2|\gamma| + 4r - 3)\Gamma t\right), \end{aligned}$$

uniformly for $t \geq 0$ and $(x, \xi) \in \mathbb{R}^{2n}$. Hence (3.11) holds for $j = r$. This ends the proof of Proposition 3.2. \square

Remark 3.7 Notice that estimate (3.11) on the derivatives of the symbols $q_j(t, x, \xi)$, $j \geq 1$, is different from the one proved in the scalar case (see [6, Theorem 1.4]). This is caused by the derivatives of the term $T(-s, \phi_\lambda^t(x, \xi))$ appearing in the expression (3.6) of $q_j(t)$ which does not exist in the scalar case. Assume that Q is classical, i.e. $Q(x, \xi) = Q_0(x, \xi)$ (as in [6]) and let us explain this difference at the level of sub-principal symbols, i.e. for $j = 1$. We have shown in Remark 3.1 that in the case where H_1 is also scalar-valued, the sub-principal symbol $q_{1,sca}(t, x, \xi)$ is given by

$$q_{1,sca}(t, x, \xi) = \int_0^t B_1(s, \phi_\lambda^{t-s}(x, \xi)) ds,$$

where B_1 is defined by (3.4). Using estimate (3.22) on the derivatives of $B_1(s, \phi_\lambda^{t-s}(x, \xi))$, one obtains

$$|\partial_{(x,\xi)}^\gamma q_{1,sca}(t, x, \xi)| \leq C_\gamma \exp(|\gamma| + 1)\Gamma|t|, \quad \forall \gamma \in \mathbb{N}^{2n}, t \in \mathbb{R}, (x, \xi) \in \mathbb{R}^{2n}, \quad (3.28)$$

which is the estimate proved in [6, Theorem 1.4]. Now, in the case where H_1 is matrix-valued, according to (3.6) (taking into account the fact that $Q_1 = 0$), $q_1(t, x, \xi)$ is given by

$$q_1(t, x, \xi) = T^{-1}(t, x, \xi) \left(\int_0^t T^{-1}(-s, \phi_\lambda^t(x, \xi)) B_1(s, \phi_\lambda^{t-s}(x, \xi)) T(-s, \phi_\lambda^t(x, \xi)) ds \right) T(t, x, \xi).$$

By going back to estimate (3.23), one sees that due to the term $T(-s, \phi_\lambda^t(x, \xi))$, when differentiating $q_1(t, x, \xi)$ $|\gamma|$ -times with respect to (x, ξ) , there is a loss of $\exp(|\gamma|\Gamma|t|)$ compared to (3.28), i.e. we have

$$\|\partial_{(x,\xi)}^\gamma q_1(t, x, \xi)\| \leq C_\gamma \exp((2|\gamma| + 1)\Gamma|t|), \quad \forall \gamma \in \mathbb{N}^{2n}, t \in \mathbb{R}, (x, \xi) \in \mathbb{R}^{2n}.$$

As pointed out in (iii) of Remark 2.4, this explains the fact that our estimate on the remainder term (2.8) is different from the one obtained in the scalar case.

3.3 Proof of Theorem 2.2

The proof of estimate (2.8) is based on estimates (3.10) and (3.11) and the control of the remainder terms in the composition formula of \hbar -pseudodifferential operators (A.3). We follow the method of Bouzouina-Robert [6].

For A, B two semiclassical symbols in suitable classes of symbols and $k \in \mathbb{N}$, we define

$$\tilde{R}_k(A, B) := i\hbar^{-(k+1)}(R_k(A, B) - R_k(B, A)), \quad (3.29)$$

where $R_k(A, B, x, \xi; \hbar) := A\#B(x, \xi; \hbar) - \sum_{j=0}^k \hbar^j (A\#B)_j$ denotes the remainder term of order k in the asymptotic expansion of the symbol $A\#B$ (see appendix A).

For $N \in \mathbb{N}$, we set

$$Q_N(t) := Q(t) - \sum_{j=0}^N \hbar^j (q_j(t))^w(x, \hbar D_x).$$

The first step in the proof of estimate (2.8) is the following lemma.

Lemma 3.8 For all $N \in \mathbb{N}$, the following estimate holds

$$\|Q_N(t)\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \leq \hbar^{N+1} \left\| \int_0^t U_H(-s) (R^{(N+1)}(t-s))^w U_H(s) ds \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} + \mathcal{O}(\hbar^{N+1}), \quad (3.30)$$

uniformly for $t \in \mathbb{R}$, with

$$R^{(N+1)}(t) := \tilde{R}_{N+1}(H, q_0(t)) + \tilde{R}_N(H, q_1(t)) + \cdots + \tilde{R}_1(H, q_N(t)) = \sum_{j=0}^N \tilde{R}_{N+1-j}(H, q_j(t)). \quad (3.31)$$

Proof. Let $N \in \mathbb{N}$ and define

$$q^{(N)}(t, x, \xi; \hbar) := \sum_{j=0}^N \hbar^j q_j(t, x, \xi).$$

According to the Cauchy problems $(\mathcal{C}_j)_{j \geq 0}$ satisfied by the symbols $q_j(t)$, for all $j \geq 0$, we have

$$\frac{d}{dt} q_j(t) = \{H, q_0(t)\}_j^* + \{H, q_1(t)\}_{j-1}^* + \{H, q_2(t)\}_{j-2}^* + \cdots + \{H, q_j(t)\}_0^*, \quad (3.32)$$

where we recall that for $0 \leq k \leq j$, $\{H, q_k(t)\}_{j-k}^*$ denotes the coefficient of \hbar^{j-k} in the asymptotic expansion of the Moyal bracket $\{H, q_k(t)\}^*$ (see appendix A). Then

$$\frac{d}{dt} q^{(N)}(t) = \sum_{j=0}^N \hbar^j \frac{d}{dt} q_j(t) = \sum_{j=0}^N \hbar^j \{H, q_0(t)\}_j^* + \hbar \sum_{j=0}^{N-1} \hbar^j \{H, q_1(t)\}_j^* + \cdots + \hbar^N \{H, q_N(t)\}_0^*.$$

Using the formula of asymptotic expansion of the Moyal bracket (A.9), we obtain

$$\{H, q^{(N)}(t)\}^* = \frac{d}{dt} q^{(N)}(t) + \hbar^{N+1} R^{(N+1)}(t), \quad (3.33)$$

with $R^{(N+1)}(t)$ defined by (3.31). A simple computation using (3.33) yields

$$\begin{aligned} \frac{d}{ds} \left(U_H(-s) Q_N(t-s) U_H(s) \right) &= U_H(-s) \left(\frac{i}{\hbar} [H^w, Q_N(t-s)] - \frac{d}{dt} Q_N(t-s) \right) U_H(s) \\ &= U_H(-s) \left(\frac{d}{dt} (q^{(N)}(t-s))^w - \frac{i}{\hbar} [H^w, (q^{(N)}(t-s))^w] \right) U_H(s) \\ &= -\hbar^{N+1} U_H(-s) (R^{(N+1)}(t-s))^w U_H(s). \end{aligned}$$

Therefore, by integrating in s and using the fact that

$$\|Q_N(0)\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} = \|Q^w(x, \hbar D_x; \hbar) - \sum_{j=0}^N \hbar^j Q_j^w(x, \hbar D_x)\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} = \mathcal{O}(\hbar^{N+1}),$$

we get

$$\|Q_N(t)\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \leq \hbar^{N+1} \left\| \int_0^t U_H(-s) (R^{(N+1)}(t-s))^w U_H(s) ds \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} + \mathcal{O}(\hbar^{N+1}),$$

uniformly for $t \in \mathbb{R}$. This ends the proof of the lemma. \square

End of the proof of Theorem 2.2.

It remains now to estimate the $\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)$ -norm of the operator $(R^{(N+1)}(t))^w$. For that, we shall employ the Calderón-Vaillancourt theorem (Theorem A.5). We shall therefore need estimates on the derivatives with respect to (x, ξ) of the symbol $R^{(N+1)}(t, x, \xi; \hbar)$.

Let $N \in \mathbb{N}$ and $0 \leq j \leq N$. We have

$$\tilde{R}_{N+1-j}(H, q_j(t)) = \tilde{R}_{N+1-j}(H_0, q_j(t)) + \tilde{R}_{N-j}(H_1, q_j(t)). \quad (3.34)$$

Let $k \in \{0, 1\}$. According to Theorem A.3 (combined with Remark A.4), for all $\gamma \in \mathbb{N}^{2n}$, there exists a constant $C = C(n, N, j, \gamma, k) > 0$ such that for all $t \in \mathbb{R}$ and all $u \in \mathbb{R}^{2n}$, we have

$$\|\partial_u^\gamma \tilde{R}_{N+1-j-k}(H_k, q_j(t); u)\| \leq C \sup_{(*)} \left(\|\partial_v^{(\alpha, \beta) + \eta} H_k(v + u)\| \|\partial_w^{(\beta, \alpha) + \kappa} q_j(t, w + u)\| \right), \quad (3.35)$$

where $\sup_{(*)}$ is the supremum under the conditions

$$(*) : \quad v, w \in \mathbb{R}^{2n}, \quad \eta, \kappa \in \mathbb{N}^{2n}; |\eta| + |\kappa| \leq 4n + 1 + |\gamma|, \quad \alpha, \beta \in \mathbb{N}^n; |\alpha| + |\beta| = N + 2 - j - k.$$

Observe first that by assumption **(A2')**, for $k \in \{0, 1\}$, for all multi-indices $((\alpha, \beta), \eta) \in \mathbb{N}^{2n} \times \mathbb{N}^{2n}$ and all $0 \leq j \leq N$ with $|\alpha| + |\beta| = N + 2 - j - k$, we have

$$\partial_{(x, \xi)}^{(\alpha, \beta) + \eta} H_k \in L^\infty(\mathbb{R}^{2n}).$$

On the other hand, using the estimates given by Proposition 3.2, for all $((\alpha, \beta), \kappa) \in \mathbb{N}^{2n} \times \mathbb{N}^{2n}$ and all $j \geq 0$, there exists $C_j = C(\alpha, \beta, \kappa, j) > 0$ such that

$$\|\partial_{(x, \xi)}^{(\beta, \alpha) + \kappa} q_0(t, x, \xi)\| \leq C_0 \exp\left((|\alpha| + |\beta| + |\kappa|)\Gamma|t|\right)$$

and for $j \geq 1$

$$\|\partial_{(x, \xi)}^{(\beta, \alpha) + \kappa} q_j(t, x, \xi)\| \leq C_j \exp\left((2(|\alpha| + |\beta| + |\kappa|) + 4j - 3)\Gamma|t|\right),$$

uniformly for $(t, x, \xi) \in \mathbb{R} \times \mathbb{R}^{2n}$.

Therefore, taking the supremum over $(*)$, there exists $C = C(\gamma, N, n, j, k) > 0$ such that for all $t \in \mathbb{R}$ and all $(x, \xi) \in \mathbb{R}^{2n}$, we have

$$\|\partial_{(x, \xi)}^\gamma \tilde{R}_{N+1-j-k}(H_k, q_j(t); x, \xi)\| \leq C \exp\left((2|\gamma| + 2N + 8n + 3 + 2j - 2k)\Gamma|t|\right).$$

Now, summing over $j = 0, \dots, N$, we get

$$\|\partial_{(x, \xi)}^\gamma R^{(N+1)}(t, x, \xi)\| \leq C_{n, N, \gamma} \exp\left((2|\gamma| + 4N + 8n + 3)\Gamma|t|\right) \quad (3.36)$$

uniformly for $t \in \mathbb{R}$ and $(x, \xi) \in \mathbb{R}^{2n}$.

Consequently, using the Calderón-Vaillancourt theorem (Theorem A.5), we deduce

$$\|(R^{(N+1)}(t))^w(x, \hbar D_x; \hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \leq C_{n, N} \exp\left((4N + \delta_n)\Gamma|t|\right),$$

uniformly for $t \in \mathbb{R}$, where δ_n is an integer depending only on the dimension n .

By going back to (3.30), we obtain

$$\begin{aligned} \|Q_N(t)\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} &\leq \hbar^{N+1} \int_0^t \|(R^{(N+1)}(t-s))^w\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} ds + \mathcal{O}(\hbar^{N+1}) \\ &\leq C_N \hbar^{N+1} \int_0^t \exp\left((4N + \delta_n)\Gamma(t-s)\right) ds + \mathcal{O}(\hbar^{N+1}) \\ &\leq C'_N \hbar^{N+1} \exp\left((4N + \delta_n)\Gamma t\right), \end{aligned}$$

uniformly for $t \geq 0$. Analogously, we prove the estimate for $t \leq 0$. This ends the proof of Theorem 2.2. □

4 General case

We now turn to the study of the general case where the principal symbol of the Hamiltonian H which generates the time evolution is no longer a scalar multiple of the identity in $M_m(\mathbb{C})$.

Let $H \in S(g)$ be an hermitian-valued semiclassical Hamiltonian satisfying **(A0)** and suppose that **(A1)** is fulfilled. We consider the time evolution of a bounded quantum observable $Q^w(x, \hbar D_x; \hbar)$ associated to a semiclassical observable $Q \in S_{\text{sc}}(1)$, given by

$$Q(t) := U_H(-t)Q^w(x, \hbar D_x; \hbar)U_H(t), \quad t \in \mathbb{R}.$$

4.1 Semiclassical projections

As already mentioned in section 2, the first step in the study of $Q(t)$ consists in the construction of the semiclassical projections associated to $H^w(x, \hbar D_x; \hbar)$.

Theorem 4.1 *Let assumption **(A1)** be satisfied. For every $1 \leq \nu \leq l$, there exists a semiclassical symbol $\tilde{P}_\nu(x, \xi; \hbar) \sim \sum_{j \geq 0} \hbar^j \tilde{P}_{\nu,j}(x, \xi)$ in $S(1, \mathbb{R}^{2n}, M_m(\mathbb{C}))$ (unique modulo $S^{-\infty}(1; \mathbb{R}^{2n}, M_m(\mathbb{C}))$) such that modulo $\mathcal{O}(\hbar^\infty)$ in $\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)$, we have*

$$\tilde{P}_\nu^w \tilde{P}_\nu^w = \tilde{P}_\nu^w \tag{4.1}$$

$$\tilde{P}_\nu^w = (\tilde{P}_\nu^w)^*, \tag{4.2}$$

$$[\tilde{P}_\nu^w, H^w] = 0. \tag{4.3}$$

$$\tilde{P}_\mu^w \tilde{P}_\nu^w = \tilde{P}_\nu^w \tilde{P}_\mu^w = 0, \quad \forall 1 \leq \mu \neq \nu \leq l, \tag{4.4}$$

$$\sum_{\nu=1}^l \tilde{P}_\nu^w = id_{L^2(\mathbb{R}^n) \otimes \mathbb{C}^m}. \tag{4.5}$$

In particular, $\tilde{P}_{\nu,0}(x, \xi) = P_{\nu,0}(x, \xi)$ is the orthogonal projector onto $\text{Ker}(H_0(x, \xi) - \lambda_\nu(x, \xi))$.

There are at least two methods of proof for this result. The first one followed by Brummelhuis and Nourrigat [5] (see also [25]) consists in solving, in the space of formal power series of \hbar , the symbolic equations corresponding to (4.1), (4.2) and (4.3),

$$\tilde{P}_\nu(x, \xi; \hbar) \# \tilde{P}_\nu(x, \xi; \hbar) \sim \tilde{P}_\nu(x, \xi; \hbar) \sim (\tilde{P}_\nu(x, \xi; \hbar))^*, \quad [\tilde{P}_\nu(x, \xi; \hbar), H(x, \xi; \hbar)]_\# \sim 0.$$

The second method due to Helffer and Sjöstrand [19] uses the Riesz projectors and the symbolic calculus of \hbar -pseudodifferential operators (see [12, ch. 8]). For the reader's convenience, we give in the appendix C an outline of the proof of Theorem 4.1 following the method in [19].

For our next purposes, it is more convenient to work with exact projections, i.e. with operators which satisfy (4.1) exactly, not only modulo $\mathcal{O}(\hbar^\infty)$ in norm $\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)$. To do this, we follow an idea from [26] (see also [25, 27]) which consists in introducing the operators

$$\mathcal{P}_\nu := \frac{i}{2\pi} \int_{|z-1|=\frac{1}{2}} (\tilde{P}_\nu^w - z)^{-1} dz, \quad 1 \leq \nu \leq l.$$

For \hbar small enough, (4.1) implies that the spectrum of \tilde{P}_ν^w is concentrated near 0 and 1 (see [26]), then \mathcal{P}_ν is well defined and satisfies

$$\mathcal{P}_\nu \mathcal{P}_\nu = \mathcal{P}_\nu = \mathcal{P}_\nu^*. \tag{4.6}$$

By a similar computation as in [26, sec. III], one gets

$$\|\mathcal{P}_\nu - \tilde{P}_\nu^w\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} = \mathcal{O}(\hbar^\infty). \tag{4.7}$$

Then, by Beals's characterization of \hbar -pseudodifferential operators (see [12, Proposition 8.3]), \mathcal{P}_ν is an \hbar -pseudodifferential operator with symbol $P_\nu(x, \xi; \hbar) \in S_{\text{sc}}(1)$, i.e. $\mathcal{P}_\nu = P_\nu^w(x, \hbar D_x; \hbar)$. Moreover, we have (see [26])

$$\|[\mathcal{P}_\nu, H^w]\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} = \mathcal{O}\left(\|[\tilde{P}_\nu^w, H^w]\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)}\right) = \mathcal{O}(\hbar^\infty). \tag{4.8}$$

4.2 Block-diagonalization

In what follows, we shall use the notation P_v^w for \mathcal{P}_v . We introduce the family of Heisenberg operators $Q_v(t)$ defined by

$$Q_v(t) := e^{\frac{it}{\hbar} P_v^w H^w P_v^w} P_v^w Q^w P_v^w e^{-\frac{it}{\hbar} P_v^w H^w P_v^w}, \quad 1 \leq v \leq l. \quad (4.9)$$

The main result of this paragraph is the following.

Proposition 4.2 *Assume that H satisfies the assumption of Theorem 4.1.*

(i) *If $Q \in \mathcal{Q}(1)$, then the following estimate holds*

$$\left\| Q(t) - \sum_{v=1}^l Q_v(t) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} = \mathcal{O}((1+|t|)\hbar^\infty), \quad (4.10)$$

uniformly for $t \in \mathbb{R}$.

(ii) *Assume that $Q_0(x, \xi) = \sum_{v=1}^l P_{v,0}(x, \xi) \tilde{Q}(x, \xi) P_{v,0}(x, \xi)$ for some $\tilde{Q} \in S(1)$. Then, we have*

$$\left\| Q(t) - \sum_{v=1}^l Q_v(t) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} = \mathcal{O}((1+|t|)\hbar), \quad (4.11)$$

uniformly for $t \in \mathbb{R}$.

The following lemma is the first step in the proof of the above proposition.

Lemma 4.3 *For all $1 \leq v \leq l$, we have*

$$e^{\frac{it}{\hbar} H^w} P_v^w = e^{\frac{it}{\hbar} P_v^w H^w P_v^w} P_v^w + \mathcal{O}(|t|\hbar^\infty), \quad (4.12)$$

uniformly for $t \in \mathbb{R}$.

Proof. Fix $v \in \{1, \dots, l\}$ and set

$$U(t) := e^{\frac{it}{\hbar} H^w} P_v^w, \quad V(t) := e^{\frac{it}{\hbar} P_v^w H^w P_v^w} P_v^w, \quad t \in \mathbb{R}.$$

Obviously, $U(t)$ satisfies

$$\begin{cases} (\hbar D_t - H^w)U(t) &= 0 \\ U(0) &= P_v^w. \end{cases} \quad (4.13)$$

Here we use the standard notation $D_t := \frac{1}{i}\partial_t$. Let us prove that $V(t)$ satisfies

$$\begin{cases} (\hbar D_t - H^w)V(t) &= I(t) \\ V(0) &= P_v^w, \end{cases} \quad (4.14)$$

with $\|I(t)\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} = \mathcal{O}(\hbar^\infty)$, uniformly for $t \in \mathbb{R}$. Put

$$R(t) := V(t) - P_v^w e^{\frac{it}{\hbar} P_v^w H^w P_v^w}.$$

Using (4.6), we get

$$\begin{aligned} \hbar D_t R(t) &= e^{\frac{it}{\hbar} P_v^w H^w P_v^w} P_v^w H^w (P_v^w)^2 - (P_v^w)^2 H^w P_v^w e^{\frac{it}{\hbar} P_v^w H^w P_v^w} \\ &= e^{\frac{it}{\hbar} P_v^w H^w P_v^w} P_v^w H^w P_v^w - P_v^w H^w P_v^w e^{\frac{it}{\hbar} P_v^w H^w P_v^w} \\ &= 0, \end{aligned} \quad (4.15)$$

which together with $R(0) = 0$ yields

$$R(t) = 0, \quad \forall t \in \mathbb{R}. \quad (4.16)$$

Now, a simple computation gives

$$\begin{aligned}
(\hbar D_t - H^w)V(t) &= (P_v^w H^w P_v^w - H^w) e^{\frac{it}{\hbar} P_v^w H^w P_v^w} P_v^w \\
&= (P_v^w H^w P_v^w - H^w) P_v^w e^{\frac{it}{\hbar} P_v^w H^w P_v^w} + (P_v^w H^w P_v^w - H^w) R(t) \\
&= (P_v^w H^w P_v^w - H^w) P_v^w e^{\frac{it}{\hbar} P_v^w H^w P_v^w} \\
&=: I(t).
\end{aligned}$$

According to (4.6), we have

$$I(t) = \left(P_v^w H^w (P_v^w)^2 - H^w P_v^w \right) e^{\frac{it}{\hbar} P_v^w H^w P_v^w} = \left(P_v^w H^w P_v^w - H^w P_v^w \right) e^{\frac{it}{\hbar} P_v^w H^w P_v^w}. \quad (4.17)$$

From (4.8), we have $P_v^w H^w = H^w P_v^w + \mathcal{O}(\hbar^\infty)$ which together with the fact that $\|P_v^w\| = \mathcal{O}(1)$ yields

$$P_v^w H^w P_v^w = H^w (P_v^w)^2 + \mathcal{O}(\hbar^\infty) = H^w P_v^w + \mathcal{O}(\hbar^\infty), \quad (4.18)$$

where in the last step, we used (4.6) again. Putting together (4.18) and (4.17), we obtain

$$\|I(t)\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} = \mathcal{O}(\hbar^\infty), \quad \text{uniformly for } t \in \mathbb{R}. \quad (4.19)$$

Now, according to Duhamel's principle, we have

$$V(t) - U(t) = \frac{1}{\hbar} \int_0^t U(t-s) \mathcal{O}(\hbar^\infty) ds,$$

which yields

$$U(t) - V(t) = \mathcal{O}(|t|\hbar^\infty), \quad \text{uniformly for } t \in \mathbb{R}. \quad (4.20)$$

This ends the proof of the lemma. \square

Turn now to the proof of Proposition 4.2.

Proof of Proposition 4.2 :

By conjugating $Q(t)$ with $\sum_{v=1}^l P_v^w(x, \hbar D_x; \hbar) = \text{id}_{L^2(\mathbb{R}^n) \otimes \mathbb{C}^m} + \mathcal{O}(\hbar^\infty)$ and using the above lemma, we get

$$\begin{aligned}
Q(t) &= \sum_{\mu, v=1}^l e^{\frac{it}{\hbar} P_\mu^w H^w P_\mu^w} P_\mu^w Q^w P_\mu^w e^{-\frac{it}{\hbar} P_v^w H^w P_v^w} + \mathcal{O}((1+|t|)\hbar^\infty) \\
&= \sum_{v=1}^l Q_v(t) + \sum_{\mu \neq v=1}^l e^{\frac{it}{\hbar} P_\mu^w H^w P_\mu^w} P_\mu^w Q^w P_\mu^w e^{-\frac{it}{\hbar} P_v^w H^w P_v^w} + \mathcal{O}((1+|t|)\hbar^\infty),
\end{aligned} \quad (4.21)$$

uniformly for $t \in \mathbb{R}$.

Passing from symbols to operators, the assumption $Q \in \mathcal{Q}(1)$ implies

$$Q^w = \sum_{v=1}^l P_v^w Q^w P_v^w + \mathcal{O}(\hbar^\infty).$$

Therefore, using that $P_\mu^w P_v^w = \mathcal{O}(\hbar^\infty)$ for $\mu \neq v$ (which follows from (4.4) and (4.7)), we deduce

$$P_\mu^w Q^w P_v^w = \mathcal{O}(\hbar^\infty), \quad \forall 1 \leq \mu \neq v \leq l. \quad (4.22)$$

Consequently, the norm $\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)$ of the second term in the right hand side of (4.21) is equal to $\mathcal{O}(\hbar^\infty)$ uniformly for $t \in \mathbb{R}$. Thus (4.10) holds.

Now, assume that $Q_0(x, \xi) = \sum_{v=1}^l P_{v,0}(x, \xi) \tilde{Q}(x, \xi) P_{v,0}(x, \xi)$, for some arbitrary $\tilde{Q} \in S(1)$. This implies that

$$[Q_0(x, \xi), P_{v,0}(x, \xi)] = 0, \quad \forall 1 \leq v \leq l, \forall (x, \xi) \in \mathbb{R}^{2n}.$$

Combining this with the fact that Q^w and P_v^w are bounded in $L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$, we get

$$[Q^w, P_v^w] = \mathcal{O}(\hbar), \quad \forall 1 \leq v \leq l,$$

which by using that $P_\mu^w P_v^w = \mathcal{O}(\hbar^\infty)$ for $\mu \neq v$ again, implies $P_v^w Q^w P_\mu^w = \mathcal{O}(\hbar)$, for all $1 \leq \mu \neq v \leq l$. Thus (4.11) holds immediately from (4.21). □

Remark 4.4 According to estimate (4.10), the study of $Q(t)$ is reduced modulo $\mathcal{O}(\hbar^\infty)$ to that of the blocks $Q_v(t)$ defined by (4.9). The main property of this reduction lies in the fact that it is preserved up to times of order $\hbar^{-\infty}$ (i.e. of order \hbar^{-k} , for all $k \in \mathbb{N}$) which in particular cover Ehrenfest type times. Thus, the problem of the construction of an asymptotic expansion in powers of \hbar for $Q(t)$ is reduced to the construction of an asymptotic expansion for each block $Q_v(t)$ defined by (4.9). This will be the object of the following paragraph.

4.3 Formal asymptotic expansion for $Q_v(t)$

From now on v will be fixed in $\{1, \dots, l\}$. We introduce the following notations for the symbols of the operators $P_v^w H^w P_v^w$ and $P_v^w Q^w P_v^w$ respectively,

$$H_v := P_v \# H \# P_v \sim \sum_{j \geq 0} \hbar^j H_{v,j} \quad (4.23)$$

$$Q_v := P_v \# Q \# P_v \sim \sum_{j \geq 0} \hbar^j Q_{v,j}. \quad (4.24)$$

Recall that by definition of $Q_v(t)$ (see (4.9)) and the fact that $(P_v^w)^j = P_v^w$, $\forall j \in \mathbb{N}$ (according to (4.6)), we have

$$(P_v^w)^j Q_v(t) (P_v^w)^j = Q_v(t), \quad \forall j \in \mathbb{N}, \forall t \in \mathbb{R}.$$

In the following this property will play an important role in the construction of an asymptotic expansion in powers of \hbar for $Q_v(t)$.

The starting point is the following Heisenberg problem

$$\begin{cases} \frac{d}{dt} Q_v(t) &= \frac{i}{\hbar} [H_v^w, Q_v(t)], \quad (t \in \mathbb{R}) \\ Q_v(t)|_{t=0} &= Q_v^w(x, \hbar D_x; \hbar), \end{cases} \quad (4.25)$$

which we rewrite at the level of symbols as

$$\begin{cases} \frac{d}{dt} q_v(t) &= \{H_v, q_v(t)\}^*, \quad (t \in \mathbb{R}) \\ q_v(t)|_{t=0} &= Q_v. \end{cases} \quad (4.26)$$

As in section 3, considering $q_v(t)$ as a formal power series of \hbar of the form $q_v(t) \sim \sum_{j \geq 0} \hbar^j q_{v,j}(t)$ and then equating equal powers of \hbar in both sides of (4.26), we derive the following Cauchy problems

$$(\mathcal{C}_{v,j}) \begin{cases} \frac{d}{dt} q_{v,j}(t) &= \{H_v, q_v(t)\}_j^* \\ q_{v,j}(t)|_{t=0} &= Q_{v,j}, \end{cases} \quad (4.27)$$

where we recall that $\{H_v, q_v(t)\}_j^* = i([H_v, q_v(t)]_\#)_j$ denotes the coefficient of \hbar^j in the asymptotic expansion of the Moyal bracket $\{H_v, q_v(t)\}^*$.

Our objective consists in looking for a solution of (4.26) of the form

$$q_v(t) \sim P_v \# \sum_{k \geq 0} \hbar^k \tilde{q}_{v,k}(t) \# P_v. \quad (4.28)$$

More explicitly, using this general form of the solution, we shall derive recursive problems for the $\tilde{q}_{v,j}(t)$. Once these problems are derived and solved, the solution $q_{v,j}(t)$ of $(\mathcal{C}_{v,j})$ can then be computed using the composition formula (A.5) from the general formula

$$q_{v,j}(t, x, \xi) = \left(P_v \# \sum_{k \geq 0} \hbar^k \tilde{q}_{v,k}(t) \# P_v \right)_j(x, \xi) = \left(P_v \# \sum_{k=0}^j \hbar^k \tilde{q}_{v,k}(t) \# P_v \right)_j(x, \xi), \quad \forall j \geq 0. \quad (4.29)$$

In particular,

$$q_{v,0}(t, x, \xi) = P_{v,0}(x, \xi) \tilde{q}_{v,0}(t, x, \xi) P_{v,0}(x, \xi). \quad (4.30)$$

Let us start by fixing the initial conditions $\tilde{q}_{v,j}(t)|_{t=0}$.

Lemma 4.5 *There exists a sequence of symbols $(\tilde{Q}_{v,k})_{k \geq 0}$ in $S(1)$ such that*

$$Q_v \sim P_v \# \sum_{k \geq 0} \hbar^k \tilde{Q}_{v,k} \# P_v$$

and

$$P_{v,0} \tilde{Q}_{v,k} P_{v,0} = \tilde{Q}_{v,k}, \quad \forall k \geq 0. \quad (4.31)$$

In particular, $\tilde{Q}_{v,0} = Q_{v,0} = P_{v,0} Q_0 P_{v,0}$.

Proof. Using the fact that $P_v \# P_v = P_v$ according to (4.6), we have

$$\begin{aligned} Q_v &= P_v \# Q \# P_v \\ &= P_v \# P_v \# Q \# P_v \# P_v \\ &\sim P_v \# (P_{v,0} Q_0 P_{v,0}) \# P_v + P_v \# \left(\sum_{j \geq 1} \hbar^j Q_{v,j} \right) \# P_v. \end{aligned}$$

Put

$$\tilde{Q}_{v,0} := P_{v,0} Q_0 P_{v,0} = Q_{v,0} \quad \text{and} \quad R_{v,0} := P_v \# \left(\sum_{j \geq 1} \hbar^j Q_{v,j} \right) \# P_v \sim \sum_{j \geq 1} \hbar^j (R_{v,0})_j.$$

We have

$$\begin{aligned} R_{v,0} &= P_v \# P_v \# R_{v,0} \# P_v \# P_v \\ &\sim \hbar P_v \# P_v \# (R_{v,0})_1 \# P_v \# P_v + P_v \# \sum_{j \geq 2} \hbar^j (R_{v,0})_j \# P_v \\ &\sim \hbar P_v \# (P_{v,0} (R_{v,0})_1 P_{v,0}) \# P_v + R_{v,1} \end{aligned}$$

where $R_{v,1} := P_v \# \sum_{j \geq 2} \hbar^j \left((P_v \# (R_{v,0})_1 \# P_v)_{j-1} + (R_{v,0})_j \right) \# P_v$. We define

$$\tilde{Q}_{v,1} := P_{v,0} (R_{v,0})_1 P_{v,0}.$$

One can iterate this procedure using at each step the fact that $R_{v,j} = P_v \# P_v \# R_{v,j} \# P_v \# P_v$ to construct the symbols $\tilde{Q}_{v,k}$ satisfying the property (4.31). The constructed symbols $\tilde{Q}_{v,k}$ are clearly in $S(1)$ since $P_v, Q \in S_{sc}(1)$. □

In view of (4.28) and the above lemma, it is thus natural to impose the following initial conditions for the $\tilde{q}_{v,j}(t)$

$$\tilde{q}_{v,j}(t)|_{t=0} = \tilde{Q}_{v,j}, \quad \forall j \geq 0. \quad (4.32)$$

Now, to derive the equations on the $\tilde{q}_{v,j}(t)$ arising from the Cauchy problems $(\mathcal{C}_{v,j})_{j \geq 0}$, we express $q_{v,j}(t)$ and $\{H_v, q_v(t)\}_j^*$ with respect to $\tilde{q}_{v,j}(t)$. For $j \geq 0$, we define

$$A_{v,j-1}(t) := P_v \# \sum_{k=0}^{j-1} \hbar^k \tilde{q}_{v,k}(t) \# P_v \quad (4.33)$$

with the convention $A_{v,-1}(t) = 0$. From (4.28), we clearly have

$$q_{v,j}(t) = P_{v,0} \tilde{q}_{v,j}(t) P_{v,0} + (A_{v,j-1}(t))_j, \quad (4.34)$$

where $(A_{v,j-1}(t))_j$ denotes the coefficient of \hbar^j in the asymptotic expansion of $A_{v,j-1}(t)$. On the other hand, we have

$$\left([H_v, q_v(t)]_\# \right)_{j+1} = [H_{v,0}, P_{v,0} \tilde{q}_{v,j+1}(t) P_{v,0}] + \left([H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_\# \right)_1 + \left([H_v, A_{v,j-1}(t)]_\# \right)_{j+1}. \quad (4.35)$$

The first term in the right hand side of the above equation vanishes since $H_{v,0} = \lambda_v P_{v,0}$. Then, putting together (4.34) and (4.35), we deduce the equation on the symbol $\tilde{q}_{v,j}(t)$ arising from $(\mathcal{C}_{v,j})$ which reads

$$\frac{d}{dt} P_{v,0} \tilde{q}_{v,j}(t) P_{v,0} = i \left([H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_\# \right)_1 + K_{v,j-1}(t), \quad (4.36)$$

where

$$K_{v,j-1}(t) := i \left([H_v, A_{v,j-1}(t)]_\# \right)_{j+1} - \frac{d}{dt} (A_{v,j-1}(t))_j. \quad (4.37)$$

Taking into account the initial conditions (4.32), we get the following Cauchy problems for $\tilde{q}_{v,j}(t)$

$$\begin{cases} \frac{d}{dt} P_{v,0} \tilde{q}_{v,j}(t) P_{v,0} &= i \left([H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_\# \right)_1 + K_{v,j-1}(t) \\ \tilde{q}_{v,j}(t)|_{t=0} &= \tilde{Q}_{v,j}. \end{cases} \quad (4.38)$$

Notice that $K_{v,j-1}(t)$ depends only on the symbols $\tilde{q}_{v,k}(t)$ with $0 \leq k \leq j-1$. ($K_{v,-1}(t) = 0$).

Proposition 4.6 *Let $j \in \mathbb{N}$. The Cauchy problem (4.38) is equivalent to the following one*

$$\begin{cases} \frac{d}{dt} P_{v,0} \tilde{q}_{v,j}(t) P_{v,0} &= \{\lambda_v, P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}\} + i [\tilde{H}_{v,1}, P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}] + K_{v,j-1}(t) \\ \tilde{q}_{v,j}(t)|_{t=0} &= \tilde{Q}_{v,j}, \end{cases} \quad (4.39)$$

where $\tilde{H}_{v,1}$ is the $(m \times m)$ hermitian-valued function defined by

$$\tilde{H}_{v,1} := \frac{\lambda_v}{2i} P_{v,0} \{P_{v,0}, P_{v,0}\} P_{v,0} - i [P_{v,0}, \{\lambda_v, P_{v,0}\}] + P_{v,0} H_{v,1} P_{v,0}. \quad (4.40)$$

To prove this proposition, we recall the following result from the appendix of [32].

Lemma 4.7 *Let $W : \mathbb{R}^{2n} \rightarrow M_m(\mathbb{C})$ be such that $[W, P_{v,0}] = 0$. We have*

$$\begin{aligned} \frac{1}{2} P_{v,0} \left(\{\lambda_v P_{v,0}, W\} - \{W, \lambda_v P_{v,0}\} \right) P_{v,0} &= \{\lambda_v, P_{v,0} W P_{v,0}\} \\ &\quad - \left[P_{v,0} W P_{v,0}, \frac{\lambda_v}{2} P_{v,0} \{P_{v,0}, P_{v,0}\} P_{v,0} + [P_{v,0}, \{\lambda_v, P_{v,0}\}] \right]. \end{aligned}$$

Proof of proposition (4.6) :

Let us start by computing $\left([H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_\# \right)_1$. We have

$$P_{v,0} \left([H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_\# \right)_1 P_{v,0} = \left([H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_\# \right)_1. \quad (4.41)$$

Indeed, the fact that $P_v \# H_v = H_v \# P_v = H_v$ (according to (4.6)) implies

$$P_v \# [H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_\# \# P_v = [H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_\#. \quad (4.42)$$

Consequently, the sub-principal symbols of the two terms in the above equation coincide. Since

$$\left([H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_{\#}\right)_0 = [H_{v,0}, P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}] = 0,$$

it follows that the sub-principal symbol of the left hand side of (4.42) is equal to

$$P_{v,0} \left([H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_{\#}\right)_1 P_{v,0}.$$

Thus we get (4.41). Using this property and formulas (A.7) and (A.8), we obtain

$$\begin{aligned} \left([H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_{\#}\right)_1 &= P_{v,0} \left([H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_{\#}\right)_1 P_{v,0} \\ &= \frac{1}{2i} P_{v,0} \left(\{H_{v,0}, P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}\} - \{P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}, H_{v,0}\} \right) P_{v,0} \\ &\quad + P_{v,0} \left([H_{v,0}, (P_v \# \tilde{q}_{v,j}(t) \# P_v)_1] + [H_{v,1}, P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}] \right) P_{v,0} \\ &= \frac{1}{2i} P_{v,0} \left(\{H_{v,0}, P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}\} - \{P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}, H_{v,0}\} \right) P_{v,0} \\ &\quad + [P_{v,0} H_{v,1} P_{v,0}, P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}], \end{aligned} \quad (4.43)$$

where in the last step we used the fact that $P_{v,0} [H_{v,0}, (P_v \# \tilde{q}_{v,j}(t) \# P_v)_1] P_{v,0} = 0$ which can be easily verified using formula (A.8). Applying Lemma 4.7 with $W = P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}$, we get

$$i \left([H_v, P_v \# \tilde{q}_{v,j}(t) \# P_v]_{\#}\right)_1 = \{\lambda_v, P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}\} + i [\tilde{H}_{v,1}, P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}],$$

where $\tilde{H}_{v,1}$ is defined by (4.40). This ends the proof of the proposition. \square

The resolution of the Cauchy problems (4.39) will be made by induction on $j \geq 0$. Let us start with $j = 0$. Since $K_{v,-1}(t) = 0$, we have

$$\begin{cases} \frac{d}{dt} P_{v,0} \tilde{q}_{v,0}(t) P_{v,0} &= \{\lambda_v, P_{v,0} \tilde{q}_{v,0}(t) P_{v,0}\} + i [\tilde{H}_{v,1}, P_{v,0} \tilde{q}_{v,0}(t) P_{v,0}] \\ \tilde{q}_{v,0}(t)|_{t=0} &= \tilde{Q}_{v,0}. \end{cases}$$

In Lemma B.2, taking into account the fact that $\tilde{Q}_{v,0} = P_{v,0} \tilde{Q}_{v,0} P_{v,0}$ (see (4.31)), we have shown that if $\tilde{q}_{v,0}(t)$ is a solution of the following problem

$$\begin{cases} \frac{d}{dt} \tilde{q}_{v,0}(t) &= \{\lambda_v, \tilde{q}_{v,0}(t)\} + i [\tilde{H}_{v,1}, \tilde{q}_{v,0}(t)] \\ \tilde{q}_{v,0}(t)|_{t=0} &= \tilde{Q}_{v,0}, \end{cases} \quad (4.44)$$

then at any time t ,

$$\tilde{q}_{v,0}(t) = P_{v,0} \tilde{q}_{v,0}(t) P_{v,0}. \quad (4.45)$$

Applying the result of Appendix B with $\Lambda = \lambda_v$ and $A = \tilde{H}_{v,1}$, we obtain the solution of (4.44) which reads

$$\tilde{q}_{v,0}(t, x, \xi) = T_v^{-1}(t, x, \xi) Q_{v,0}(\phi_v^t(x, \xi)) T_v(t, x, \xi), \quad (4.46)$$

where T_v in the unitary $(m \times m)$ matrix-valued function solution of the system

$$\frac{d}{dt} T_v(t, x, \xi) = -i \tilde{H}_{v,1}(\phi_v^t(x, \xi)) T_v(t, x, \xi), \quad T_v(0, x, \xi) = I_m. \quad (4.47)$$

Let us now assume that we have solved (4.39) until the order $j - 1$, i.e. we have constructed the symbols $\tilde{q}_{v,k}(t)$ for $k \in \{0, \dots, j - 1\}$ and that they satisfy

$$\tilde{q}_{v,k}(t) = P_{v,0} \tilde{q}_{v,k}(t) P_{v,0}, \quad \forall k \in \{0, \dots, j - 1\}.$$

We are going to solve (4.39) at the order j and check that the solution $\tilde{q}_{v,j}(t)$ satisfies

$$\tilde{q}_{v,j}(t) = P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}.$$

To apply Lemma B.2, we have to prove that

$$P_{v,0} K_{v,j-1}(t) P_{v,0} = K_{v,j-1}(t). \quad (4.48)$$

Recall that $K_{v,j-1}(t)$ defined by (4.37) is the j -th term (i.e. the coefficient of \hbar^j) of the symbol

$$E_{v,j-1}(t) := \frac{i}{\hbar} [H_v, A_{v,j-1}(t)]_{\#} - \frac{d}{dt} A_{v,j-1}(t).$$

In the following, we say that $B \sim \sum_{k \geq 0} \hbar^k B_k$ belongs to $S(\hbar^j)$ if $B_k = 0$ for all $k < j$.

We claim that

$$E_{v,j-1}(t) \in S(\hbar^j). \quad (4.49)$$

This will be proven below. Due to (4.33) and (4.23), we have

$$P_v \# E_{v,j-1}(t) \# P_v = E_{v,j-1}(t).$$

By equating the j -th terms in both sides using (4.49) we get (4.48). Taking into account (4.31) and (4.48), according to Lemma B.2, if $\tilde{q}_{v,j}(t)$ is a solution of the following problem

$$\begin{cases} \frac{d}{dt} \tilde{q}_{v,j}(t) &= \{\lambda_v, \tilde{q}_{v,j}(t)\} + i[\tilde{H}_{v,1}, \tilde{q}_{v,j}(t)] + K_{v,j-1}(t) \\ \tilde{q}_{v,j}(t)|_{t=0} &= \tilde{Q}_{v,j}, \end{cases} \quad (4.50)$$

then

$$\tilde{q}_{v,j}(t) = P_{v,0} \tilde{q}_{v,j}(t) P_{v,0}, \quad \forall t \in \mathbb{R}.$$

To solve (4.50), we apply the result of Appendix B again with $\Lambda = \lambda_v$, $A = \tilde{H}_{v,1}$ and $B(t) = K_{v,j-1}(t)$. The solution reads

$$\tilde{q}_{v,j}(t, x, \xi) = T_v^{-1}(t, x, \xi) \left(\tilde{Q}_{v,j}(\phi_v^t(x, \xi)) + \int_0^t W_{v,j}(t, s, x, \xi) ds \right) T_v(t, x, \xi), \quad (4.51)$$

with

$$W_{v,j}(t, s, x, \xi) := T_v^{-1}(-s, \phi_v^t(x, \xi)) K_{v,j-1}(s, \phi_v^{t-s}(x, \xi)) T_v(-s, \phi_v^t(x, \xi)),$$

where T_v is given by the system (4.47).

It remains now to prove the claim (4.49) by induction on j . For $j = 1$, we have

$$\begin{aligned} (E_{v,0}(t))_0 &= i([H_v, A_{v,0}(t)]_{\#})_1 - \frac{d}{dt} (A_{v,0}(t))_0 \\ &= i([H_v, P_v \# \tilde{q}_{v,0}(t) \# P_v]_{\#})_1 - \frac{d}{dt} P_{v,0} \tilde{q}_{v,0}(t) P_{v,0} \\ &= 0, \end{aligned}$$

since it is the equation satisfied by $\tilde{q}_{v,0}(t)$ (see (4.38)). Thus $E_{v,0}(t) \in S(\hbar)$.

We assume that $E_{v,j-2}(t) \in S(\hbar^{j-1})$ and let us prove (4.49). Using that

$$A_{v,j-1}(t) = A_{v,j-2}(t) + \hbar^{j-1} P_v \# \tilde{q}_{v,j-1}(t) \# P_v$$

we get

$$\begin{aligned} E_{v,j-1}(t) &= E_{v,j-2}(t) - \hbar^{j-1} \frac{d}{dt} P_v \# \tilde{q}_{v,j-1}(t) \# P_v + i \hbar^{j-2} [H_v, P_v \# \tilde{q}_{v,j-1}(t) \# P_v]_{\#} \\ &= E_{v,j-2}(t) - \hbar^{j-1} \frac{d}{dt} P_{v,0} \tilde{q}_{v,j-1}(t) P_{v,0} + S(\hbar^j) + i \hbar^{j-1} ([H_v, P_v \# \tilde{q}_{v,j-1}(t) \# P_v]_{\#})_1 + S(\hbar^j) \\ &= E_{v,j-2}(t) - \hbar^{j-1} \left(\frac{d}{dt} P_{v,0} \tilde{q}_{v,j-1}(t) P_{v,0} - i([H_v, P_v \# \tilde{q}_{v,j-1}(t) \# P_v]_{\#})_1 \right) + S(\hbar^j). \end{aligned} \quad (4.52)$$

Notice that to pass from the first to the second equality, we have used the fact that

$$[H_v, P_v \# \tilde{q}_{v,j-1}(t) \# P_v]_{\#} \in S(\hbar)$$

since as it was already point out in (4.35) its principal symbol vanishes.

On the other hand, combining the definition of $K_{v,j-2}(t)$ which is $K_{v,j-2}(t) = (E_{v,j-2}(t))_{j-1}$ and the induction hypothesis $E_{v,j-2}(t) \in S(\hbar^{j-1})$, we get

$$E_{v,j-2}(t) = \hbar^{j-1} K_{v,j-2}(t) + S(\hbar^j).$$

By going back to (4.52), we obtain

$$E_{v,j-1}(t) = \hbar^{j-1} \left(K_{v,j-2}(t) - \frac{d}{dt} P_{v,0} \tilde{q}_{v,j-1}(t) P_{v,0} + i([H_v, P_v \# \tilde{q}_{v,j-1}(t) \# P_v]_{\#})_1 \right) + S(\hbar^j).$$

The first term in the right hand side of the above equation vanishes since it is exactly the equation satisfied by $\tilde{q}_{v,j-1}(t)$ (see (4.38)). Thus, we proved that $E_{v,j-1}(t) \in S(\hbar^j)$. This ends the proof of the claim. □

Summing up, we hence have solved the Cauchy problems (4.38) for all $j \geq 0$. The solutions $(\tilde{q}_{v,j}(t))_{j \geq 0}$ are given by formula (4.51). In particular, $\tilde{q}_{v,0}(t)$ is given by (4.46). As already mentioned in the beginning of this paragraph, the solutions $q_{v,j}(t)$ of the Cauchy problems $(\mathcal{C}_{v,j})_{j \geq 0}$ can then be computed using the composition formula (A.5) from the general formula (4.29). In particular, the principal symbol $q_{v,0}(t)$ is given by (4.30).

4.4 Uniform estimates and proofs of Theorem 2.5 and Corollary 2.7

This section is devoted to the proofs of Theorem 2.5 and Corollary 2.7. Since the techniques of the proofs are close to those used in the above section, we shall omit some details.

As in section 3, we start by estimating the derivatives of the constructed symbols $q_{v,j}(t)$, $j \geq 0$.

Proposition 4.8 *Assume (A1) and (A2) and let $1 \leq \nu \leq l$. For all $\gamma \in \mathbb{N}^{2n}$, for all $j \geq 0$, there exists $C_{\gamma,\nu,j} > 0$ such that for all $t \in \mathbb{R}$ and all $(x, \xi) \in \mathbb{R}^{2n}$, we have*

$$\|\partial_{(x,\xi)}^{\gamma} q_{v,0}(t, x, \xi)\| \leq C_{\gamma,\nu,0} \exp(|\gamma| \Gamma_{\nu} |t|), \quad (4.53)$$

and for $j \geq 1$,

$$\|\partial_{(x,\xi)}^{\gamma} q_{v,j}(t, x, \xi)\| \leq C_{\gamma,\nu,j} \exp((2|\gamma| + 4j - 2) \Gamma_{\nu} |t|), \quad (4.54)$$

where Γ_{ν} is defined by (2.12).

Similarly to the proof of Proposition 3.2, the proof of the above proposition is based on the following lemmas which give estimates on the derivatives of the Hamiltonian flows ϕ_v^t generated by the eigenvalues λ_v and the matrix-valued function T_v defined in (4.47).

From now on we fix $\nu \in \{1, \dots, l\}$.

Lemma 4.9 *We assume that*

$$\partial_{(x,\xi)}^{\gamma} H_0 \in L^{\infty}(\mathbb{R}^{2n}), \quad \text{for } |\gamma| \geq 2. \quad (4.55)$$

Then, for all $\gamma \in \mathbb{N}^{2n} \setminus \{0\}$, there exists $C_{\nu,\gamma} > 0$ such that for all $t \in \mathbb{R}$ and all $(x, \xi) \in \mathbb{R}^{2n}$,

$$\|\partial_{(x,\xi)}^{\gamma} \phi_v^t(x, \xi)\| \leq C_{\nu,\gamma} \exp(|\gamma| \Gamma_{\nu} |t|). \quad (4.56)$$

Proof. According to inequality (C.1), (4.55) implies that $\partial_{(x,\xi)}^\gamma \lambda_v \in L^\infty(\mathbb{R}^{2n})$, for $|\gamma| \geq 2$. Thus estimate (4.56) can be proved in the same manner as in Lemma 3.4 (see [6, Lemma 2.2]). \square

We turn now to the estimation of the derivatives of T_v solution of the system (4.47).

Lemma 4.10 *Let assumptions (A1) and (A2) be satisfied. For all $\gamma \in \mathbb{N}^{2n} \setminus \{0\}$ there exists a constant $C_{v,\gamma} > 0$ (independent of $t \in \mathbb{R}$ and $(x, \xi) \in \mathbb{R}^{2n}$) such that*

$$\|\partial_{(x,\xi)}^\gamma T_v(t, x, \xi)\| \leq C_{v,\gamma} \exp(|\gamma| \Gamma_v |t|). \quad (4.57)$$

Furthermore, the same estimate holds for $T_v^{-1}(t, x, \xi)$.

Proof. We recall the expression of the $(m \times m)$ hermitian-valued function $\tilde{H}_{v,1}$ defined in (4.40)

$$\tilde{H}_{v,1} = P_{v,0} H_{v,1} P_{v,0} - i[P_{v,0}, \{\lambda_v, P_{v,0}\}] - \frac{i}{2} \lambda_v P_{v,0} \{P_{v,0}, P_{v,0}\} P_{v,0} := I_v^{(1)} + I_v^{(2)} + I_v^{(3)}.$$

We claim that under assumptions (A1) and (A2), we have $\tilde{H}_{v,1} \in S(1)$. Then, estimate (4.57) can be proved by applying exactly the same method as in the proof of Lemma 3.5.

To prove the claim let us start by computing $H_{v,1}$. From (4.8), we have $H_{v,1} := (P_v \# H \# P_v)_1 = (P_v \# H)_1$. Then, using formula (A.6), we obtain

$$H_{v,1} = \frac{1}{2i} \{P_{v,0}, H_0\} + P_{v,0} H_1 + P_{v,1} H_0. \quad (4.58)$$

It follows that

$$I_v^{(1)} = \frac{1}{2i} P_{v,0} \{P_{v,0}, H_0\} P_{v,0} + P_{v,0} H_1 P_{v,0} + \lambda_v P_{v,0} P_{v,1} P_{v,0}.$$

Computing $P_{v,1}$ using formula (A.6) and multiplying from both sides by $P_{v,0}$, we get

$$\lambda_v P_{v,0} P_{v,1} P_{v,0} = \frac{i}{2} \lambda_v P_{v,0} \{P_{v,0}, P_{v,0}\} P_{v,0} = -I_v^{(3)}.$$

Consequently,

$$\tilde{H}_{v,1} = \frac{1}{2i} P_{v,0} \{P_{v,0}, H_0\} P_{v,0} + P_{v,0} H_1 P_{v,0} - i[P_{v,0}, \{\lambda_v, P_{v,0}\}]. \quad (4.59)$$

Using assumption (A2) and Lemma C.1, we clearly see that $\tilde{H}_{v,1} \in S(1)$. This ends the proof of the lemma. \square

Remark 4.11 *As in Lemma 3.6, combining (4.56) and (4.57) and using the Faà Di Bruno formula (3.12), we get the following estimate on the derivatives of $T_v(s, \phi_\lambda^t(x, \xi))$: for all $\gamma \in \mathbb{N}^{2n}$, there exists $C_{v,\gamma} > 0$ such that*

$$\|\partial_{(x,\xi)}^\gamma (T_v(s, \phi_\lambda^t(x, \xi)))\| \leq C_{v,\gamma} \exp(|\gamma| \Gamma_v (|t| + |s|)), \quad (4.60)$$

uniformly for $t, s \in \mathbb{R}$ and $(x, \xi) \in \mathbb{R}^{2n}$. The same estimate remains valid for $T_v^{-1}(s, \phi_\lambda^t(x, \xi))$.

We end our series of Lemmas by the following one where we control the derivatives of the symbols $(H_{v,j})_{j \geq 0}$.

Lemma 4.12 *Under assumptions (A1) and (A2), for all $j \geq 0$ and $\gamma \in \mathbb{N}^{2n}$ with $|\gamma| + j \geq 1$, we have*

$$\partial_{(x,\xi)}^\gamma H_{v,j} \in L^\infty(\mathbb{R}^{2n}). \quad (4.61)$$

Proof. From the proof of Lemma C.1, one verify that by combining condition (2.11) and assumption (A2), we get

$$\|\partial_{(x,\xi)}^\gamma P_{v,0}(x,\xi)\| \leq C_\gamma g^{-1}(x,\xi), \quad \forall |\gamma| \geq 1. \quad (4.62)$$

Thus, since $H_{v,0} = \lambda_v P_{v,0}$, then (4.61) for $j = 0$ follows immediately from (4.62) and inequality (C.1).

Now, for $j \geq 1$, from the composition formula (A.5) we have

$$\begin{aligned} H_{v,j} = (P_v \# H)_j &= \sum_{|\alpha|+|\beta|+k+p=j} \gamma(\alpha, \beta) P_{v,k}^{(\beta)} H_p^{(\alpha)} \\ &= \sum_{|\alpha|+|\beta|+k=j} \gamma(\alpha, \beta) P_{v,k}^{(\beta)} H_0^{(\alpha)} + \sum_{|\alpha|+|\beta|+k=j-1} \gamma(\alpha, \beta) P_{v,k}^{(\beta)} H_1^{(\alpha)}. \end{aligned}$$

According to Lemma C.2, we have $P_{v,k} \in S(g^{-k})$, for all $k \geq 1$. Then, using (A2), we obtain (4.61) for all $j \geq 1$. \square

Now, we are in position to prove Proposition 4.8.

Proof of Proposition 4.8 :

For $j = 0$, estimate (4.53) is a direct consequence of estimates (4.56) and (4.57).

Let us prove estimate (4.54). In the following, when it is not precised, all constants $C_\gamma > 0$ are uniform with respect to $t \in \mathbb{R}$ and $(x, \xi) \in \mathbb{R}^{2n}$.

We start by proving (4.54) for the derivatives of $\tilde{q}_{v,j}(t)$, $j \geq 1$, i.e.

$$\|\partial_{(x,\xi)}^\gamma \tilde{q}_{v,j}(t, x, \xi)\| \leq C_{\gamma,v,j} \exp\left((2|\gamma| + 4j - 2)\Gamma_v |t|\right), \quad \forall \gamma \in \mathbb{N}^{2n}. \quad (4.63)$$

We proceed by induction with respect to j . Recall the expression of $\tilde{q}_{v,1}(t)$

$$\tilde{q}_{v,1}(t, x, \xi) = T_v^{-1}(t, x, \xi) \left(\tilde{Q}_{v,1}(\phi_v^t(x, \xi)) + \int_0^t W_{v,1}(t, s, x, \xi) ds \right) T_v(t, x, \xi),$$

where

$$\begin{aligned} W_{v,1}(t, s, x, \xi) &= T_v^{-1}(-s, \phi_v^t(x, \xi)) K_{v,0}(s, \phi_v^{t-s}(x, \xi)) T_v(-s, \phi_v^t(x, \xi)) \\ K_{v,0}(t, x, \xi) &= i \left([H_v(x, \xi; \hbar), A_{v,0}(t, x, \xi; \hbar)]_\# \right)_2 - \frac{d}{dt} (A_{v,0}(t, x, \xi; \hbar))_1 \end{aligned}$$

and

$$A_{v,0}(t, x, \xi; \hbar) = (P_v \# \tilde{q}_{v,0}(t) \# P_v)(x, \xi; \hbar). \quad (4.64)$$

Let us estimating the derivatives of $K_{v,0}(t, x, \xi)$. Since $H_v \# P_v = P_v \# H_v = H_v$, it follows that

$$[H_v, A_{v,0}(t)]_\# = P_v \# [H_v, \tilde{q}_{v,0}(t)]_\# \# P_v.$$

From this equation, using the composition formula (A.5), we see that $\left([H_v, A_{v,0}(t)]_\#\right)_2$ is a finite linear combination of terms depending on the symbols $P_{v,k}$, $H_{v,j}$, $\tilde{q}_{v,0}(t)$ and theirs derivatives with at most a derivative of order 2 (with respect to (x, ξ)) of $\tilde{q}_{v,0}(t, x, \xi)$. The term $H_{v,0}$ appears only in the commutator $[H_{v,0}, \tilde{q}_{v,0}(t)]$ which vanishes since $P_{v,0} \tilde{q}_{v,0}(t) P_{v,0} = \tilde{q}_{v,0}(t)$ (see (4.45)). Consequently, using estimate (4.53), the fact that $P_{v,k} \in S(1)$ and Lemma 4.12, we obtain

$$\left\| \partial_{(x,\xi)}^\gamma \left([H_v, A_{v,0}(t)]_\# \right)_2 (x, \xi) \right\| \leq C_\gamma \exp((|\gamma| + 2)\Gamma_v |t|), \quad \forall \gamma \in \mathbb{N}^{2n}. \quad (4.65)$$

On the other hand, from (4.64) we have

$$\frac{d}{dt} (A_{v,0}(t))_1 = \left(P_v \# \frac{d}{dt} \tilde{q}_{v,0}(t) \# P_v \right)_1.$$

Since $\tilde{q}_{v,0}(t)$ satisfies equation (4.44), i.e.

$$\frac{d}{dt}\tilde{q}_{v,0}(t) = \{\lambda_v, \tilde{q}_{v,0}(t)\} + i[\tilde{H}_{v,1}, \tilde{q}_{v,0}(t)],$$

it follows from estimate (4.53) again, assumption **(A2)** and the fact that $\tilde{H}_{v,1} \in S(1)$ (see the proof of Lemma 4.10) that for all $\gamma \in \mathbb{N}^{2n}$, there exists $C_\gamma > 0$ independent of $t \in \mathbb{R}$ and $(x, \xi) \in \mathbb{R}^{2n}$ such that

$$\left\| \partial_{(x,\xi)}^\gamma \left(\frac{d}{dt}(A_{v,0}(t))_1 \right) (x, \xi) \right\| \leq C_\gamma \exp((|\gamma| + 2)\Gamma_v |t|). \quad (4.66)$$

Putting together (4.65) and (4.66), we obtain

$$\left\| \partial_{(x,\xi)}^\gamma K_{v,0}(t, x, \xi) \right\| \leq C_\gamma \exp((|\gamma| + 2)\Gamma_v |t|), \quad \forall \gamma \in \mathbb{N}^{2n}.$$

As in the proof of Proposition 3.2, using the above estimate, estimate (4.56) on the derivatives of the flow ϕ_v^t , estimate (4.57) on the derivatives of $T_v(t, x, \xi)$ and the Faà Di Bruno formula (3.12), we get

$$\left\| \partial_{(x,\xi)}^\gamma \tilde{q}_{v,1}(t, x, \xi) \right\| \leq C_\gamma \exp((2|\gamma| + 2)\Gamma_v |t|), \quad \forall \gamma \in \mathbb{N}^{2n}.$$

Thus we proved that (4.63) for $j = 1$.

Let us now assume that $\tilde{q}_{v,k}(t, x, \xi)$ satisfies (4.63) for $k \in \{1, \dots, r-1\}$. Recall the expression of $\tilde{q}_{v,r}(t)$

$$\tilde{q}_{v,r}(t, x, \xi) = T_v^{-1}(t, x, \xi) \left(\tilde{Q}_{v,r}(\phi_v^t(x, \xi)) + \int_0^t W_{v,r}(t, s, x, \xi) ds \right) T_v(t, x, \xi),$$

where

$$\begin{aligned} W_{v,r}(t, s, x, \xi) &= T_v^{-1}(-s, \phi_v^t(x, \xi)) K_{v,r-1}(s, \phi_v^{t-s}(x, \xi)) T_v(-s, \phi_v^t(x, \xi)) \\ K_{v,r-1}(t, x, \xi) &= i \left([H_v(x, \xi; \hbar), A_{v,r-1}(t, x, \xi; \hbar)]_\# \right)_{r+1} - \frac{d}{dt}(A_{v,r-1}(t, x, \xi; \hbar))_r \end{aligned}$$

and

$$A_{v,r-1}(t, x, \xi; \hbar) = (P_v \# \sum_{k=0}^{r-1} \hbar^k \tilde{q}_{v,k}(t) \# P_v)(x, \xi; \hbar).$$

As above, we have

$$[H_v, A_{v,r-1}(t)]_\# = P_v \# \left[H_v, \sum_{k=0}^{r-1} \hbar^k \tilde{q}_{v,k}(t) \right]_\# \# P_v$$

which yields

$$\left([H_v, A_{v,r-1}(t)]_\# \right)_{r+1} = \sum_{k=0}^{r-1} \left(P_v \# [H_v, \tilde{q}_{v,k}(t)]_\# \# P_v \right)_{r+1-k}.$$

Again, using the composition formula (A.5), we see that for all $k \in \{0, \dots, r-1\}$, $(P_v \# [H_v, \tilde{q}_{v,k}(t)]_\# \# P_v)_{r+1-k}$ depends at most on a derivative of order $r+1-k$ of $\tilde{q}_{v,k}(t)$ (and on the derivatives of $H_{v,j}$ and $P_{v,l}$). Consequently, using the induction hypothesis, we get

$$\left\| \partial_{(x,\xi)}^\gamma \left([H_v, A_{v,r-1}(t)]_\# \right)_{r+1} (x, \xi) \right\| \leq C_{\gamma,r} \exp((2|\gamma| + 4r - 2)\Gamma_v |t|), \quad \forall \gamma \in \mathbb{N}^{2n}. \quad (4.67)$$

Since $\frac{d}{dt}A_{v,r-1}(t)$ depends on $\frac{d}{dt}\tilde{q}_{v,k}(t)$, $k \in \{0, \dots, r-1\}$, which satisfy equations (4.50), it follows that to estimate the derivatives with respect to (x, ξ) of $(\frac{d}{dt}A_{v,r-1}(t))_r$, one first needs estimates on the derivatives of $K_{v,k}(t)$ with $k \in \{0, \dots, r-2\}$. This can be made by induction on k and we get that $(\frac{d}{dt}A_{v,r-1}(t))_r$ satisfies estimate (4.67). Consequently, we obtain

$$\left\| \partial_{(x,\xi)}^\gamma K_{v,r-1}(t, x, \xi) \right\| \leq C_{\gamma,r} \exp((2|\gamma| + 4r - 2)\Gamma_v |t|), \quad \forall \gamma \in \mathbb{N}^{2n}.$$

We conclude as in the proof of Proposition 3.2 using estimates (4.56), (4.60) and Leibniz formula. Hence,

$$\|\partial_{(x,\xi)}^\gamma \tilde{q}_{v,j}(t, x, \xi)\| \leq C_{\gamma,v,j} \exp\left((2|\gamma| + 4j - 2)\Gamma_v |t|\right), \quad \forall \gamma \in \mathbb{R}^{2n}, \forall j \geq 1, \quad (4.68)$$

uniformly for $t \in \mathbb{R}$ and $(x, \xi) \in \mathbb{R}^{2n}$. This ends the proof of (4.63).

Turn now to the proof of estimate (4.54). Let $j \geq 1$. According to the general form of the solution (4.29), we have

$$q_{v,j}(t, x, \xi) = \left(P_v \# \tilde{q}_{v,0}(t) \# P_v\right)_j(x, \xi) + \left(P_v \# \tilde{q}_{v,1}(t) \# P_v\right)_{j-1}(x, \xi) + \cdots + \left(P_v \# \tilde{q}_{v,j}(t) \# P_v\right)_0(x, \xi).$$

By the composition formula (A.5), each term $\left(P_v \# \tilde{q}_{v,k}(t) \# P_v\right)_{j-k}(x, \xi)$, $k \in \{0, \dots, j\}$, in the above sum is a finite linear combination of terms depending on $P_{v,l}(x, \xi)$, $\tilde{q}_{v,k}(t, x, \xi)$ and their derivatives (with respect to (x, ξ)) with at most a derivative of order $j - k$ of $\tilde{q}_{v,k}(t, x, \xi)$. Then, using (4.63) and the fact that $P_{v,l} \in S(1)$ for all $l \geq 0$, we deduce that for all $1 \leq k \leq j$ and $\gamma \in \mathbb{R}^{2n}$, we have

$$\|\partial_{(x,\xi)}^\gamma \left(P_v \# \tilde{q}_{v,k}(t) \# P_v\right)_{j-k}(x, \xi)\| \leq C_{j,k,\gamma,v} \exp\left((2|\gamma| + 2(j+k) + 2)\Gamma_v |t|\right). \quad (4.69)$$

Taking the supremum over $k \in \{1, \dots, j\}$, we get (4.54). This ends the proof of Proposition 4.8.

□

4.4.1 Proofs of Theorem 2.5 and Corollary 2.7

Proof of Theorem 2.5 :

The starting point is the same as in the proof of Theorem 2.2. Set

$$U_{H_v}(t) := e^{-\frac{it}{\hbar} H_v^w} = e^{-\frac{it}{\hbar} P_v^w H^w P_v^w}, \quad t \in \mathbb{R}.$$

For $N \in \mathbb{N}$, let $Q_v^{(N)}(t)$ be the remainder term of order N in the asymptotic expansion of $Q_v(t)$, i.e.

$$Q_v^{(N)}(t) := Q_v(t) - \sum_{j=0}^N \hbar^j (q_{v,j}(t))^w(x, \hbar D_x).$$

Lemma 4.13 Fix $1 \leq v \leq l$. For all $N \in \mathbb{N}$, the following estimate holds

$$\left\| Q_v^{(N)}(t) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \leq \hbar^{N+1} \left\| \int_0^t U_{H_v}(-s) (R_v^{(N+1)}(t-s))^w U_{H_v}(s) ds \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} + \mathcal{O}(\hbar^{N+1}),$$

uniformly for $t \in \mathbb{R}$, where

$$R_v^{(N+1)}(t) := \tilde{R}_{N+1}(H_v, q_{v,0}(t)) + \tilde{R}_N(H_v, q_{v,1}(t)) + \cdots + \tilde{R}_1(H_v, q_{v,N}(t)). \quad (4.70)$$

We recall that the notation $\tilde{R}_k(A, B)$ is introduced in (3.29).

For $N \in \mathbb{N}$, we set

$$Q^{(N)}(t) := Q(t) - \sum_{j=0}^N \hbar^j \sum_{v=1}^l (q_{v,j}(t))^w(x, \hbar D_x).$$

Using Lemma 4.13 and Proposition 4.2 (i), we obtain

$$\begin{aligned} \left\| Q^{(N)}(t) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} &\leq \sum_{v=1}^l \left\| Q_v^{(N)}(t) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} + \left\| Q(t) - \sum_{v=1}^l Q_v(t) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \\ &\leq l \hbar^{N+1} \sup_{1 \leq v \leq l} \left\| \int_0^t U_{H_v}(-s) (R_v^{(N+1)}(t-s))^w U_{H_v}(s) ds \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} + \mathcal{O}(\hbar^{N+1}) \\ &\quad + \mathcal{O}((1+|t|)\hbar^\infty), \end{aligned} \quad (4.71)$$

uniformly for $t \in \mathbb{R}$.

As in the end of the proof of Theorem 2.2, using the estimates on the symbols $q_{v,j}(t)$ given by Proposition 4.8, Theorem A.3 and the Calderón-Vaillancourt theorem (Theorem A.5), we prove the following estimate

$$\left\| \left(R_v^{(N+1)}(t) \right)^w(x, \hbar D_x; \hbar) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \leq C_{v,n,N} \exp\left((4N + \tilde{\delta}_n) \Gamma_v |t|\right),$$

uniformly for $t \in \mathbb{R}$, where $\tilde{\delta}_n$ is an integer depending only on the dimension n . We conclude as in the end of the proof of Theorem 2.2. □

Proof of Corollary 2.7 :

Let $Q(x, \xi) \sim \sum_{j \geq 0} \hbar^j Q_j(x, \xi)$ in $S(1)$ and assume that there exists $\tilde{Q} \in S(1)$ such that

$$Q_0(x, \xi) = \sum_{v=1}^l P_{v,0}(x, \xi) \tilde{Q}(x, \xi) P_{v,0}(x, \xi).$$

According to Proposition 4.2 (ii), we have

$$\left\| Q(t) - \sum_{v=1}^l Q_v(t) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} = \mathcal{O}((1 + |t|)\hbar), \quad \text{uniformly for } t \in \mathbb{R}.$$

Thus by rewriting (4.71) for $N = 0$ and using Lemma 4.13, we get

$$\begin{aligned} \left\| Q^{(0)}(t) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} &\leq \sum_{v=1}^l \left\| Q_v^{(0)}(t) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} + \left\| Q(t) - \sum_{v=1}^l Q_v(t) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \\ &\leq l \hbar \sup_{1 \leq v \leq l} \left\| \int_0^t U_{H_v}(-s) \left(R_v^{(1)}(t-s) \right)^w U_{H_v}(s) ds \right\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} + \mathcal{O}(\hbar) + \mathcal{O}((1 + |t|)\hbar), \end{aligned}$$

uniformly for $t \in \mathbb{R}$. We conclude as above. □

We end this section by the following remark concerning an application of the results of this paper.

Remark 4.14 (Application) Consider the matrix semiclassical Schrödinger operator in $L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$

$$P(\hbar) := -\hbar^2 \Delta \otimes I_m + V(x), \tag{4.72}$$

where V is a $(m \times m)$ hermitian-valued potential satisfying the following long-range assumption

(S1). There exists an hermitian matrix $V_\infty \in M_m(\mathbb{C})$ and a constant $\delta > 0$ such that for all $\alpha \in \mathbb{N}^n$,

$$\left\| \partial_x^\alpha (V(x) - V_\infty) \right\| \leq C_\alpha \langle x \rangle^{-\delta - |\alpha|}, \quad \forall x \in \mathbb{R}^n.$$

The limiting absorption principle (see e.g. [16]) ensures that the boundary values of the resolvent of $P(\hbar)$,

$$(P(\hbar) - (E \pm i0))^{-1} := \lim_{\varepsilon \searrow 0} (P(\hbar) - (E \pm i\varepsilon))^{-1}$$

exists as bounded operators from $L_s^2(\mathbb{R}^n) \otimes \mathbb{C}^m$ to $L_{-s}^2(\mathbb{R}^n) \otimes \mathbb{C}^m$ for any $s > \frac{1}{2}$ and E outside the pure point spectrum of $P(\hbar)$. Here $L_s^2(\mathbb{R}^n) \otimes \mathbb{C}^m$ denotes the space of \mathbb{C}^m -valued functions defined on \mathbb{R}^n such that $x \mapsto \langle x \rangle^s f(x)$ belongs to $L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$.

In the scalar case, i.e. when $m = 1$, (resp. the matrix-valued case without crossings eigenvalues), a well known result is a bounds $\mathcal{O}(\hbar^{-1})$ on these boundary values near non-trapping energies for the Hamiltonian $p(x, \xi) := |\xi|^2 + V(x)$ (resp. for the eigenvalues of $p(x, \xi) := |\xi|^2 I_m + V(x)$). We refer to [31] (resp. [21]) for the proofs of these results. In the case of trapped energies, using the results of Bouzouina-Robert [6], Bony, Burq and Ramond [4] proved a lower bound of the type $\hbar^{-1} \log(\hbar^{-1})$ on the boundary values of the resolvent of scalar Schrödinger operators. According to the remark after Theorem 2 in [4] and our main results (Theorem 2.5 and Corollary 2.6), we can obtain the same lower bound for the boundary values of the operator (4.72). The detailed proof will appear elsewhere.

A Review of semiclassical pseudodifferential calculus for matrix valued symbols

In this section we recall some notions and results about the semiclassical pseudodifferential calculus in the context of operators with matrix-valued symbols. These results are well known in the case of scalar-valued symbols and we refer to [12, ch. 7-9] and [35, ch. 4] for more details.

The set of Weyl operators with symbols in the classes $S(g)$ introduced in section 2 is stable under the operator multiplication. Let σ be the canonical symplectic form on \mathbb{R}^{2n}

$$\sigma(x, \xi; y, \zeta) := \langle J(x, \xi), (y, \zeta) \rangle, \quad J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \forall (x, \xi, y, \zeta) \in \mathbb{R}^{4n}. \quad (\text{A.1})$$

More precisely, we have the following well known result (see [30, 35])

Theorem A.1 *Let g_1, g_2 be two order functions on \mathbb{R}^{2n} . The map*

$$\begin{aligned} S(g_1) \times S(g_2) &\longrightarrow S(g_1 g_2) \\ (P, Q) &\longmapsto P \# Q \end{aligned}$$

where $P \# Q$ is defined by :

$$P \# Q(x, \xi) := e^{\frac{i\hbar}{2} \sigma(D_x, D_\xi; D_y, D_\eta)} (P(x, \xi) Q(y, \eta))|_{(x, \xi) = (y, \eta)}, \quad (\text{A.2})$$

is a bilinear continuous map in the topology generated by the semi-norms associated to (2.2) and we have

$$(P \# Q)^w(x, \hbar D_x) = P^w(x, \hbar D_x) \circ Q^w(x, \hbar D_x),$$

as operators mapping $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}^m$ to $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}^m$. The symbol $P \# Q$, called the Moyal product of P, Q , admits the following asymptotic expansion in powers of \hbar

$$P \# Q(x, \xi) \sim \sum_{j \geq 0} \frac{\hbar^j}{j!} \left(\frac{i}{2} \sigma(D_x, D_\xi; D_y, D_\eta) \right)^j (P(x, \xi) Q(y, \eta))|_{(x, \xi) = (y, \eta)} \quad \text{in } S(g_1 g_2). \quad (\text{A.3})$$

Furthermore, if $P(x, \xi; \hbar) \sim \sum_{j \geq 0} \hbar^j P_j(x, \xi)$ in $S(g_1)$ and $Q(x, \xi; \hbar) \sim \sum_{j \geq 0} \hbar^j Q_j(x, \xi)$ in $S(g_2)$ are two semiclassical symbols, then $P \# Q$ is again a semiclassical symbol and we have

$$P \# Q(x, \xi; \hbar) \sim \sum_{j \geq 0} \hbar^j (P \# Q)_j(x, \xi) \quad \text{in } S(g_1 g_2), \quad (\text{A.4})$$

where for all $j \geq 0$,

$$(P \# Q)_j(x, \xi) := \sum_{|\alpha| + |\beta| + k + l = j} \gamma(\alpha, \beta) P_k^{(\beta)}(x, \xi) Q_l^{(\alpha)}(x, \xi), \quad (\text{A.5})$$

with $\gamma(\alpha, \beta) := \frac{(-1)^{|\beta|}}{(2i)^{|\alpha| + |\beta|} \alpha! \beta!}$. In particular, the principal symbol and the sub-principal symbol of $P \# Q$ are respectively given by

$$(P \# Q)_0 = P_0 Q_0, \quad (P \# Q)_1 = \frac{1}{2i} \{P_0, Q_0\} + P_0 Q_1 + P_1 Q_0. \quad (\text{A.6})$$

In the following remark we collect some useful identities which can be easily computed using (A.6).

Remark A.2 *We recall that the Moyal commutator $[P, Q]_\#$ of P and Q is defined as $[P, Q]_\# := P \# Q - Q \# P$. For $P \sim \sum_{j \geq 0} \hbar^j P_j$, $Q \sim \sum_{j \geq 0} \hbar^j Q_j$ and $C \sim \sum_{j \geq 0} \hbar^j C_j$ three semiclassical matrix-valued symbols, we have*

$$([P, Q]_\#)_0 = [P_0, Q_0], \quad ([P, Q]_\#)_1 = \frac{1}{2i} (\{P_0, Q_0\} - \{Q_0, P_0\}) + [P_0, Q_1] + [P_1, Q_0]. \quad (\text{A.7})$$

$$(P \# Q \# C)_0 = P_0 Q_0 C_0, \quad (P \# Q \# C)_1 = \frac{1}{2i} \{P_0 Q_0, C_0\} + P_0 Q_0 C_1 + \frac{1}{2i} \{P_0, Q_0\} C_0 + P_0 Q_1 C_0 + P_1 Q_0 C_0. \quad (\text{A.8})$$

The Moyal bracket

Let $P \sim \sum_{j \geq 0} \hbar^j P_j$ in $S(g_1)$ and $Q \sim \sum_{j \geq 0} \hbar^j Q_j$ in $S(g_2)$ be two matrix-valued semiclassical symbols. The Moyal bracket of P, Q denoted $\{P, Q\}^*$ is defined as the Weyl symbol of $\frac{i}{\hbar}[P^w, Q^w]$. By means of the Moyal product it can be written as

$$\{P, Q\}^* := \frac{i}{\hbar}[P, Q]_{\#} = \frac{i}{\hbar}(P \# Q - Q \# P).$$

If the principal symbols P_0 and Q_0 commute, i.e. if $[P_0, Q_0] = 0$, then using the rule of asymptotic expansion of the Moyal product of symbols (formula A.4), one can expand $\{P, Q\}^*$ in a power series of \hbar and gets

$$\{P, Q\}^* \sim \sum_{j \geq 0} \hbar^j \{P, Q\}_j^* \quad \text{in } S(g_1 g_2), \quad (\text{A.9})$$

with $\{P, Q\}_j^* = i([P, Q]_{\#})_{j+1} = i((P \# Q)_{j+1} - (Q \# P)_{j+1})$, for all $j \geq 0$.

Let $N \geq 1$. The remainder term of order $N-1$ in the asymptotic expansion (A.9) can be expressed by means of the remainder terms in the asymptotic expansions of $P \# Q$ and $Q \# P$. More precisely, we have

$$\{P, Q\}^* - \sum_{j=0}^{N-1} \hbar^j \{P, Q\}_j^* = i \hbar^{-1} (R_N(P, Q) - R_N(Q, P)), \quad (\text{A.10})$$

where

$$R_N(P, Q; x, \xi; \hbar) := P \# Q(x, \xi) - \sum_{j=0}^N \hbar^j (P \# Q)_j(x, \xi) \quad (\text{A.11})$$

denotes the remainder term of order N in the asymptotic expansion of $P \# Q$.

Remainder estimate in the composition formula

In [6, Theorem A.1], Bouzouina and Robert established the following estimate on the derivatives of $R_N(P, Q)$ in the case of scalar-valued symbols. This result remains true without any change in the case of matrix-valued symbols.

Theorem A.3 *There exists a constant $K_n > 0$ such that for every integer $\kappa \geq 4n$ and every $s > 4n$, there exists $\tau_{n, \kappa, s} > 0$ such that for every $P, Q \in \mathcal{S}(\mathbb{R}^{2n}) \otimes M_m(\mathbb{C})$ we have :*

For every $N \geq 1$ and every $\gamma \in \mathbb{N}^{2n}$, the following estimate holds for every $u \in \mathbb{R}^{2n}$

$$\begin{aligned} \|\partial_u^\gamma R_N(P, Q; u; \hbar)\| &\leq \hbar^{N+1} \tau_{n, \kappa, s} K_n^{N+|\gamma|} (N!)^{-1} \\ &\times \sup_{\substack{v, w \in \mathbb{R}^{2n} \\ \mu, v \in \mathbb{N}^{2n}; |\mu| + |v| \leq \kappa + |\gamma| \\ \alpha, \beta \in \mathbb{N}^n; |\alpha| + |\beta| = N+1}} \left(\langle (v, w) \rangle^{s-\kappa} \|\partial_v^{(\alpha, \beta) + \mu} P(v + u)\| \|\partial_w^{(\beta, \alpha) + v} Q(w + u)\| \right). \end{aligned} \quad (\text{A.12})$$

Remark A.4 *As it was shown in [6], using the fact that $\mathcal{S}(\mathbb{R}^{2n}) \otimes M_m(\mathbb{C})$ is dense in $S(\langle u \rangle^a; \mathbb{R}^{2n}, M_m(\mathbb{C}))$, $a \in \mathbb{R}$, for the topology of the Fréchet spaces $S(\langle u \rangle^{a+\varepsilon}; \mathbb{R}^{2n}, M_m(\mathbb{C}))$, for all $\varepsilon > 0$, Theorem A.3 can be extended to symbols $P \in S(\langle u \rangle^a; \mathbb{R}^{2n}, M_m(\mathbb{C}))$ and $Q \in S(\langle u \rangle^b; \mathbb{R}^{2n}, M_m(\mathbb{C}))$, with $a, b \in \mathbb{R}$ such that $\kappa - s \geq a + b$ to get a finite right hand side in (A.12).*

We end this background by the following well known result (see [35, ch. 4]).

Theorem A.5 (Calderón-Vaillancourt) *There exists an integer k_n and a constant $C_n > 0$ such that if $Q \in S(1)$ then $Q^w(x, \hbar D_x; \hbar) : L^2(\mathbb{R}^n) \otimes \mathbb{C}^m \rightarrow L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$ is bounded and we have*

$$\|Q^w(x, \hbar D_x; \hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^m)} \leq C_n \sup_{|\alpha| + |\beta| \leq k_n} \hbar^{\frac{|\alpha| + |\beta|}{2}} \|\partial_\xi^\alpha \partial_x^\beta Q\|_{L^\infty(\mathbb{R}^{2n})}. \quad (\text{A.13})$$

B Cauchy problem

Let $\Lambda \in C^\infty(\mathbb{R}^{2n}; \mathbb{R})$, $A \in C^\infty(\mathbb{R}^{2n}) \otimes M_m(\mathbb{C})$ hermitian-valued and $B \in C^\infty(\mathbb{R} \times \mathbb{R}^{2n}) \otimes M_m(\mathbb{C})$. In this paragraph, we give the general solution of the following Cauchy problem

$$\begin{cases} \frac{d}{dt}\psi(t, x, \xi) &= \{\Lambda, \psi(t, \cdot, \cdot)\}(x, \xi) + i[A(x, \xi), \psi(t, x, \xi)] + B(t, x, \xi) \\ \psi(t, x, \xi)|_{t=0} &= \psi_0(x, \xi), \end{cases} \quad (\text{B.1})$$

which arises when we solve the Cauchy problems (3.1) and (4.38) in sections 3 and 4, respectively. We assume that the flow $\phi_\Lambda^t(x, \xi)$ exists globally on \mathbb{R} for all $(x, \xi) \in \mathbb{R}^{2n}$ since it is the case for ϕ_λ^t and ϕ_v^t (see section 2).

We introduce the $(m \times m)$ matrix-valued function T solution of the following system

$$\frac{d}{dt}T(t, x, \xi) = -iA(\phi_\Lambda^t(x, \xi))T(t, x, \xi), \quad T(0, x, \xi) = I_m. \quad (\text{B.2})$$

The following lemma was proved in [5, Proposition 4].

Lemma B.1 *The matrix $T(t, x, \xi)$ is unitary and we have*

$$T(-t, \phi_\Lambda^t(x, \xi)) = T^{-1}(t, x, \xi), \quad \forall t \in \mathbb{R}, (x, \xi) \in \mathbb{R}^{2n}. \quad (\text{B.3})$$

Notice that in [5], the quantity $\Gamma(t, x, \xi) = T(-t, \phi_\Lambda^t(x, \xi))$ was considered instead of T . The equation satisfied by T^{-1} reads

$$\frac{d}{dt}T^{-1}(t, x, \xi) = iT^{-1}(t, x, \xi)A(\phi_\Lambda^t(x, \xi)). \quad (\text{B.4})$$

A simple computation using (B.2) and (B.4) yields

$$\begin{aligned} \frac{d}{dt}\left(T^{-1}(-t, x, \xi)\psi(t, \phi_\Lambda^{-t}(x, \xi))T(-t, x, \xi)\right) = \\ T^{-1}(-t, x, \xi)\left(\frac{d}{dt}\psi(t, \phi_\Lambda^{-t}(x, \xi)) - \{\Lambda, \psi(t)\} \circ \phi_\Lambda^{-t}(x, \xi) - i[A, \psi(t)] \circ \phi_\Lambda^{-t}(x, \xi)\right)T(-t, x, \xi). \end{aligned}$$

Consequently, equation (B.1) is equivalent to the following one

$$\frac{d}{dt}\left(T^{-1}(-t, x, \xi)\psi(t, \phi_\Lambda^{-t}(x, \xi))T(-t, x, \xi)\right) = T^{-1}(-t, x, \xi)B(t, \phi_\Lambda^{-t}(x, \xi))T(-t, x, \xi).$$

Therefore

$$\psi(t, \phi_\Lambda^{-t}(x, \xi)) = T(-t, x, \xi)\left(\psi_0(x, \xi) + \int_0^t T^{-1}(-s, x, \xi)B(s, \phi_\Lambda^{-s}(x, \xi))T(-s, x, \xi) ds\right)T^{-1}(-t, x, \xi). \quad (\text{B.5})$$

Using Lemma B.1, we obtain the solution of (B.1) which reads

$$\psi(t, x, \xi) = T^{-1}(t, x, \xi)\left(\psi_0(\phi_\Lambda^t(x, \xi)) + \int_0^t T^{-1}(-s, \phi_\Lambda^t(x, \xi))B(s, \phi_\Lambda^{t-s}(x, \xi))T(-s, \phi_\Lambda^t(x, \xi)) ds\right)T(t, x, \xi).$$

□

The following lemma is used in the proof of Proposition 4.6. Similar result was announced in the appendix of [32] (see equation (A.22) therein).

Lemma B.2 *Consider the Cauchy problem (B.1) with $\Lambda = \lambda_v$ and $A = \tilde{H}_{v,1}$ defined by (4.40). We assume that ψ_0 and $B(t)$ satisfy*

$$\psi_0 = P_{v,0}\psi_0P_{v,0} \quad \text{and} \quad B(t) = P_{v,0}B(t)P_{v,0}, \quad \forall t \in \mathbb{R}.$$

Then the solution $\psi(t)$ satisfies

$$\psi(t) = P_{v,0}\psi(t)P_{v,0}, \quad \forall t \in \mathbb{R}.$$

Proof. Put $\bar{P}_{v,0} := I_m - P_{v,0}$. We shall prove that

$$\bar{P}_{v,0}\psi(t) = 0 \quad \text{and} \quad \psi(t)\bar{P}_{v,0} = 0, \quad \forall t \in \mathbb{R}.$$

We have

$$\begin{aligned} \frac{d}{dt}\bar{P}_{v,0}\psi(t) &= \bar{P}_{v,0}\{\lambda_v, \psi(t)\} - \bar{P}_{v,0}[\psi(t), i\tilde{H}_{v,1}] \\ &= \{\lambda_v, \bar{P}_{v,0}\psi(t)\} - \{\lambda_v, \bar{P}_{v,0}\}\psi(t) - \bar{P}_{v,0}[\psi(t), i\tilde{H}_{v,1}] \\ &= \{\lambda_v, \bar{P}_{v,0}\psi(t)\} - \{\lambda_v, \bar{P}_{v,0}\}\psi(t) + i\bar{P}_{v,0}\tilde{H}_{v,1}\psi(t) - i\bar{P}_{v,0}\psi(t)\tilde{H}_{v,1} \\ &= \{\lambda_v, \bar{P}_{v,0}\psi(t)\} - \{\lambda_v, \bar{P}_{v,0}\}\psi(t) + i\bar{P}_{v,0}\tilde{H}_{v,1}\psi(t) + \mathcal{O}(\bar{P}_{v,0}\psi(t)), \end{aligned}$$

where we used the fact that $\tilde{H}_{v,1} \in S(1)$ (see the proof of Lemma 4.10). According to the definition of $\tilde{H}_{v,1}$ we have

$$i\bar{P}_{v,0}\tilde{H}_{v,1}\psi(t) = \bar{P}_{v,0}[P_{v,0}, \{\lambda_v, P_{v,0}\}]\psi(t) = -\bar{P}_{v,0}\{\lambda_v, P_{v,0}\}P_{v,0}\psi(t). \quad (\text{B.6})$$

Next, multiplying the obvious equality $\{\lambda_v, P_{v,0}\} = \{\lambda_v, P_{v,0}^2\} = \{\lambda_v, P_{v,0}\}P_{v,0} + P_{v,0}\{\lambda_v, P_{v,0}\}$ on the left and right by $P_{v,0}$, gives $P_{v,0}\{\lambda_v, P_{v,0}\}P_{v,0} = 2P_{v,0}\{\lambda_v, P_{v,0}\}P_{v,0}$ and then $P_{v,0}\{\lambda_v, P_{v,0}\}P_{v,0} = 0$. Combining this with (B.6), we obtain

$$i\bar{P}_{v,0}\tilde{H}_{v,1}\psi(t) = \{\lambda_v, \bar{P}_{v,0}\}P_{v,0}\psi(t).$$

Therefore, we have

$$\begin{aligned} \frac{d}{dt}\bar{P}_{v,0}\psi(t) &= \{\lambda_v, \bar{P}_{v,0}\psi(t)\} - \{\lambda_v, \bar{P}_{v,0}\}\psi(t) + \{\lambda_v, \bar{P}_{v,0}\}P_{v,0}\psi(t) + \mathcal{O}(\bar{P}_{v,0}\psi(t)) \\ &= \{\lambda_v, \bar{P}_{v,0}\psi(t)\} - \{\lambda_v, \bar{P}_{v,0}\}\bar{P}_{v,0}\psi(t) + \mathcal{O}(\bar{P}_{v,0}\psi(t)), \end{aligned}$$

which by using the fact that $\{\lambda_v, \bar{P}_{v,0}\} = \mathcal{O}(1)$ (which follows from assumption (A2) and Lemma C.1) gives

$$\frac{d}{dt}\bar{P}_{v,0}\psi(t) = \{\lambda_v, \bar{P}_{v,0}\psi(t)\} + \mathcal{O}(\bar{P}_{v,0}\psi(t)).$$

Put $g(t, x, \xi) := \bar{P}_{v,0}(x, \xi)\psi(t, x, \xi)$ and $f(t, x, \xi) := g(t, \phi_v^{-t}(x, \xi))$. Taking into account the fact that $f(0) = g(0) = \bar{P}_{v,0}\psi_0 = 0$ (since $\psi_0 = P_{v,0}\psi_0 P_{v,0}$ by hypothesis), we have

$$\frac{d}{dt}f(t, x, \xi) = \mathcal{O}(f(t, x, \xi)), \quad f(0) = 0.$$

Consequently, using Gronwall Lemma, we get

$$f(t) = 0, \quad \forall t \in \mathbb{R}.$$

Hence

$$\bar{P}_{v,0}\psi(t) = 0, \quad \forall t \in \mathbb{R}.$$

The same arguments show that $\psi(t)\bar{P}_{v,0} = 0$, for all $t \in \mathbb{R}$. This ends the proof of the lemma. \square

C Semiclassical projections

In this appendix, we prove that under assumption (A1), λ_v and $P_{v,0}$ belong to nice classes of symbols, for all $1 \leq v \leq l$, and we give an idea of the proof of Theorem 4.1. For more details we refer to [27] and the original paper [19] (see also [3]).

Lemma C.1 Fix $1 \leq v \leq l$. Under assumption (A1), $P_{v,0} \in S(1)$ and for all $\gamma \in \mathbb{N}^{2n}$, there exists $C_\gamma > 0$ such that

$$|\partial_{(x,\xi)}^\gamma \lambda_v(x, \xi)| \leq C_\gamma \|\partial_{(x,\xi)}^\gamma H_0(x, \xi)\|, \quad \forall (x, \xi) \in \mathbb{R}^{2n}. \quad (\text{C.1})$$

In particular, $\lambda_v \in S(g)$.

Proof. Let $v \in \{1, \dots, l\}$. Let $\varepsilon(x, \xi) > 0$ be such that

$$0 < \frac{\rho}{2} g(x, \xi) \leq \varepsilon(x, \xi) \leq \frac{1}{2} \min_{1 \leq \mu \neq v \leq l} |\lambda_\mu(x, \xi) - \lambda_v(x, \xi)|. \quad (\text{C.2})$$

Put

$$\gamma_v(x, \xi) := \{z \in \mathbb{C}; |z - \lambda_v(x, \xi)| = \varepsilon(x, \xi)\}, \quad (\text{C.3})$$

and

$$P_{v,0}(x, \xi) = \frac{i}{2\pi} \int_{\gamma_v(x, \xi)} (H_0(x, \xi) - z)^{-1} dz.$$

By the Cauchy theorem, we see that a small variation of the contour $\gamma_v(x, \xi)$ does not change $P_{v,0}(x, \xi)$. Let $z \in \gamma_v(x, \xi)$. According to (C.2), $(H_0(x, \xi) - z)^{-1}$ exists for all $(x, \xi) \in \mathbb{R}^{2n}$ and since $H_0(x, \xi)$ is hermitian it follows that

$$\|(H_0(x, \xi) - z)^{-1}\| \leq \frac{1}{\text{dist}(z, \sigma(H_0(x, \xi)))} \leq \frac{2}{\rho} g^{-1}(x, \xi), \quad (\text{C.4})$$

where $\sigma(H_0(x, \xi)) := \{\lambda_1(x, \xi), \dots, \lambda_l(x, \xi)\}$. Combining (C.4) and the fact that $H_0 \in S(g)$, one sees that $P_{v,0} \in S(1)$. For $\gamma = 0$, (C.1) is obvious. Taking the derivatives of the equation $(P_{v,0}(x, \xi))^2 = P_{v,0}(x, \xi)$, we obtain

$$P_{v,0}(x, \xi) \partial_{(x, \xi)}^\gamma P_{v,0}(x, \xi) P_{v,0}(x, \xi) = 0, \quad \forall \gamma \in \mathbb{N}^{2n} \setminus \{0\}. \quad (\text{C.5})$$

Now, by differentiating successively the equation $H_0(x, \xi) P_{v,0}(x, \xi) = \lambda_v(x, \xi) P_{v,0}(x, \xi)$ using (C.5) and the fact that $P_{v,0} \in S(1)$, one gets (C.1) for all $\gamma \in \mathbb{N}^{2n} \setminus \{0\}$. \square

Outline of the proof of Theorem 4.1 :

Fix $1 \leq v \leq l$ and let $\gamma_v(x, \xi)$ be the contour defined in (C.3). According to (C.4), for all $z \in \gamma_v(x, \xi)$, $(H_0(x, \xi) - z)$ is elliptic, i.e. $(H_0(x, \xi) - z)^{-1} \in S(g^{-1})$. By the composition formula (A.3), we have

$$\begin{aligned} (H(x, \xi; \hbar) - z) \# (H_0(x, \xi) - z)^{-1} &= (H_0(x, \xi) - z) \# (H_0(x, \xi) - z)^{-1} + \hbar H_1(x, \xi) \# (H_0(x, \xi) - z)^{-1} \\ &= I_m - \hbar r(x, \xi, z; \hbar), \end{aligned} \quad (\text{C.6})$$

with $r \in S(1)$, uniformly for $z \in \gamma_v(x, \xi)$. Consequently, using the symbolic calculus of \hbar -pseudodifferential operators (see [12, ch. 8]), we can construct a parametrix $B \in S(g^{-1})$ such that for $z \in \gamma_v(x, \xi)$,

$$B(x, \xi, z; \hbar) \sim \sum_{j \geq 0} \hbar^j B_j(x, \xi, z) \text{ in } S(g^{-1}), \text{ with } B_0(x, \xi, z) = (H_0(x, \xi) - z)^{-1}, \quad (\text{C.7})$$

and

$$B(x, \xi, z; \hbar) \# (H(x, \xi; \hbar) - z) \sim (H(x, \xi; \hbar) - z) \# B(x, \xi, z; \hbar) \sim I_m, \quad (\text{C.8})$$

in $S(1)$. The above formula implies that for $z, \tilde{z} \in \gamma_v(x, \xi)$

$$\begin{aligned} (H(x, \xi; \hbar) - z) \# [B(x, \xi, z; \hbar) - B(x, \xi, \tilde{z}; \hbar)] \# (H(x, \xi; \hbar) - \tilde{z}) &\sim (z - \tilde{z}) I_m, \\ (H(x, \xi; \hbar) - z) \# B(x, \xi, z; \hbar) \# B(x, \xi, \tilde{z}; \hbar) \# (H(x, \xi; \hbar) - \tilde{z}) &\sim I_m, \end{aligned}$$

which yields

$$B(x, \xi, z; \hbar) - B(x, \xi, \tilde{z}; \hbar) \sim (z - \tilde{z}) B(x, \xi, z; \hbar) \# B(x, \xi, \tilde{z}; \hbar). \quad (\text{C.9})$$

Put

$$\tilde{P}_v(x, \xi; \hbar) := \frac{i}{2\pi} \int_{\gamma_v(x, \xi)} B(x, \xi, z; \hbar) dz \sim \frac{i}{2\pi} \sum_{j \geq 0} \hbar^j \int_{\gamma_v(x, \xi)} B_j(x, \xi, z) dz. \quad (\text{C.10})$$

By construction of $\gamma_v(x, \xi)$ and $B(x, \xi, z; \hbar)$, we easily see that $\tilde{P}_v(x, \xi; \hbar) \in S(1)$.

Let us start by proving (4.1). As we already pointed out in the above proof, by the Cauchy theorem, a small variation of the contour $\gamma_v(x, \xi)$ does not change $\tilde{P}_v(x, \xi; \hbar)$. Let $\tilde{\gamma}_v(x, \xi)$ be a simple closed

contour with the same properties than $\gamma_v(x, \xi)$ contained inside $\gamma_v(x, \xi)$. Clearly, (C.9) remains true for $z \in \gamma_v(x, \xi)$ and $\tilde{z} \in \tilde{\gamma}_v(x, \xi)$.

Using (C.9), we obtain

$$\begin{aligned} \tilde{P}_v(x, \xi; \hbar) \# \tilde{P}_v(x, \xi; \hbar) &= \left(\frac{i}{2\pi} \right)^2 \int_{\gamma_v(x, \xi)} \int_{\tilde{\gamma}_v(x, \xi)} B(x, \xi, z; \hbar) \# B(x, \xi, \tilde{z}; \hbar) dz d\tilde{z} \\ &\sim \left(\frac{i}{2\pi} \right)^2 \int_{\gamma_v(x, \xi)} \int_{\tilde{\gamma}_v(x, \xi)} \left(\frac{1}{z - \tilde{z}} B(x, \xi, z; \hbar) + \frac{1}{\tilde{z} - z} B(x, \xi, \tilde{z}; \hbar) \right) dz d\tilde{z} \\ &=: I_1 + I_2 \end{aligned} \quad (\text{C.11})$$

where

$$\begin{aligned} I_1 &:= \left(\frac{i}{2\pi} \right)^2 \int_{\gamma_v(x, \xi)} \left(\int_{\tilde{\gamma}_v(x, \xi)} \frac{1}{z - \tilde{z}} d\tilde{z} \right) B(x, \xi, z; \hbar) dz = 0 \\ I_2 &:= \left(\frac{i}{2\pi} \right)^2 \int_{\tilde{\gamma}_v(x, \xi)} \left(\int_{\gamma_v(x, \xi)} \frac{1}{\tilde{z} - z} dz \right) B(x, \xi, \tilde{z}; \hbar) d\tilde{z} = \frac{i}{2\pi} \int_{\tilde{\gamma}_v(x, \xi)} B(x, \xi, \tilde{z}; \hbar) d\tilde{z}. \end{aligned}$$

This gives (4.1). Property (4.2) follows immediately from the selfadjointness of $H^w(x, \hbar D_x; \hbar)$ while (4.3) is a consequence of (C.8).

In order to prove (4.4), we consider two contours $\gamma_v(x, \xi)$ and $\gamma_\mu(x, \xi)$ such that $\text{dist}(\gamma_v(x, \xi), \gamma_\mu(x, \xi)) \geq c > 0$ and we repeat the same computation as in C.11 with $\gamma_\mu(x, \xi)$ instead of $\tilde{\gamma}_v(x, \xi)$. In this case $I_1 = I_2 = 0$.

Formula (4.5), follows from the construction of $\tilde{P}_v(x, \xi; \hbar)$ which yields

$$\sum_{v=1}^l \tilde{P}_v(x, \xi; \hbar) \sim I_m.$$

□

The following lemma is needed in the proof of Lemma 4.12. Put

$$\tilde{P}_{v,j}(x, \xi) := \frac{i}{2\pi} \int_{\gamma_v(x, \xi)} B_j(x, \xi, z) dz, \quad j \geq 0.$$

Lemma C.2 *Under assumptions (A1) and (A2), we have*

$$\tilde{P}_{v,j} \in S(g^{-j}), \quad \forall j \geq 0.$$

Proof. For all $j \geq 0$, $B_j(x, \xi, z)$ is given by (see equation (8.11) in [12])

$$B_j(x, \xi, z) = (H_0(x, \xi) - z)^{-1} \#^j r := (H_0(x, \xi) - z)^{-1} \# r \# r \cdots \# r,$$

with $\#$ repeated j -times. The symbol r is defined in (C.6), more precisely

$$\begin{aligned} r(x, \xi, z; \hbar) &= \frac{1}{\hbar} (I_m - (H(x, \xi; \hbar) - z) \# (H_0(x, \xi) - z)^{-1}) \\ &= \frac{1}{\hbar} (I_m - (H_0(x, \xi) - z) \# (H_0(x, \xi) - z)^{-1}) - H_1(x, \xi) \# (H_0(x, \xi) - z)^{-1}. \end{aligned}$$

Since $(H_0(x, \xi) - z)^{-1} \in S(g^{-1})$ according to (C.4), it follows from assumption (A2) and the composition formula (A.3) that $r \in S(g^{-1})$. Then, for all $j \geq 0$

$$(H_0(x, \xi) - z)^{-1} \#^j r \in S(g^{-(j+1)}).$$

Consequently, $\tilde{P}_{v,j} \in S(g^{-j})$, for all $j \geq 0$. □

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